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This thesis aims to investigate various methods for computing globally defined, periodic solutions of the $3 \times 3$ compressible Euler equations. The methods implemented for computing such solutions all start with a chosen solution of the linear wave equation. We use this to perturb a stationary solution, forming an approximate solution to the Euler equations which is to then be refined to the desired period solution to the original nonlinear problem. This is done by evolving the approximate solution nonlinearly via a mixed spectral finite-difference method. A correction to the approximate solution is then found using some quasi-Newton method. This process is then applied iteratively until a desired accuracy is reached.

## 1 The Euler equations and the problems with sound

Before Euler, the field of fluid dynamics was only just emerging. Compared to the contributions his predecessors, Newton and Bernoulli, Euler's contributions to the field were unparalleled. In the 1750s, Euler wrote the governing equations for a fluid. At the time, this was limited to the momentum equation and the continuity equation. Written in a Lagrangian frame, these are

$$\begin{cases} \partial_t u + \nabla p = 0 \\ \partial_t v - \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

where $u$ is the velocity, $p$ the pressure, and $v$ the specific volume. However, the lack of a third equation leaves this system underdetermined unless the fluid is taken to be incompressible, $\nabla \cdot u = 0$, or barotropic, $p = p(v)$.

Laplace added the third equation, that describing conservation of energy, in 1816. With this addition we have the compressible Euler equations, the focus of this dissertation,

$$\begin{cases} \partial_t u + \nabla p = 0 \\ \partial_t v - \nabla \cdot u = 0 \\ \partial_t (\frac{1}{2} u^2 + e) + \nabla \cdot (up) = 0. \end{cases} \tag{1.2}$$

Euler tried to develop a theory of sound from just the two equations in (1.1). Trying to investigate the vibrations of a string, Euler linearized these two equations, obtaining the wave equation that D'Alembert had formulated and solved. Based on this, it seemed one could explain any sound they heard as an appropriate combination of pure tones. However, a number of nineteenth-century mathematicians would notice a problem with this. In his 1848 "*On the velocity of sound*" Challis showed an apparent contradiction that comes from using the exact solution of a traveling wave that Poisson derived in "*On the theory of sound* in 1808. Namely, Challis found that the velocity should become multivalued in finite time [3]. Stokes quickly published a response to Challis, "*On a difficulty in the theory of sound*", in which he derives the breaking time, the time

at which one has infinite gradients and cannot find a single-valued solution. The prospect of some kind of discontinuous surface as a solution was noted to have strange implications, Stokes only able to suggest that while some strange kind of discontinuous motion may not exist in nature, it might just be a consequence of considering a ideal fluid, similarly absent in nature. Stokes nevertheless continued investigating the mathematical side of such solutions, deriving the jump conditions for the mass and momentum [9].

Riemann expanded on this in 1860 with "*On the propagation of plane air waves of finite amplitude*", showing that for the isentropic case compressions necessarily form shock waves and deriving bounds for the speed of these shock waves [8]. Even jumping ahead decades to Glimm and Lax or later, the general belief was still that this was also true in the nonisentropic case, that compressions always form shocks even with nonconstant entropy. And so there has long been a disconnect between the mathematical understanding and the physical reality of sound waves. That is, we notice a distinct lack of shock waves in the course of a typical conversation. The problem was thermodynamics needed to mature further still in order to reach a better understanding of the problem [11]. Since we instead hear sound waves that are smooth, and notably shock free, throughout every moment of our day, one might expect there to exist smooth solutions of (1.2) if we only we could use the thermodynamics that was lacked by some of these mathematicians, but which we have access to, to ensure the problem is as accurate a representation of reality as possible, rather than an idealized setting we will never actually encounter.

This dissertation follows the recent work of Temple and Young that proved the existence of smooth, periodic, oscillatory solutions of the $3 \times 3$ compressible Euler equations (1.2). In "*The nonlinear theory of sound*," they answer this long-standing problem in the theory of sound by showing that while the pure tone solutions of the wave equation may form shocks on their own, they can be perturbed to smooth, periodic solutions [10]. In particular, they focus on the theoretical proof of the existence of solutions in a 1-dimensional spatial domain that satisfy initial conditions

$$p(0,\cdot) \quad \text{even}, \qquad u(0,\cdot) = 0,$$

boundary conditions

$$d_x p(\cdot,0) = d_x p(\cdot,T) = 0, \qquad u(\cdot,0) = u(\cdot,T) = 0,$$

and periodicity condition

$$\tfrac{\mathcal{I}-\mathcal{R}}{2}\mathcal{S}^{\frac{T}{4}} p(l,\cdot) = 0, \qquad \tfrac{\mathcal{I}-\mathcal{R}}{2}\mathcal{S}^{\frac{T}{4}} u(l,\cdot) = 0.$$

Here $\mathcal{I}$ denotes the identity, $\mathcal{S}^{t_0}$ a shift in time by $t_0$, and $\mathcal{R}$ reflection in time,

$$\mathcal{S}^{t_0} y(x,t) = y(x,t-t_0), \qquad \mathcal{R}y(x,t) = y(x,-t).$$

We first formulate and computationally implement a method for computing these solutions based on this proof, but due to the specifics of the proof, it is not immediately clear how to handle higher spatial dimensions.

In order to be able to find solutions in higher spatial dimensions, this dissertation also constructs a modified method for computing periodic solutions that easily lends itself to higher spatial dimensions. Computations on 2-dimensional spatial domains are certainly expensive, even if not prohibitively so, as seen in this dissertation's inclusion of solutions computed on multiple 2-dimensional domains, even solutions corresponding to audible frequencies at quite high volumes. But then in three dimensions, we not only face the curse of dimensionality but also far greater computational costs. Nevertheless, we seek to justify the ideas about a nonlinear theory of sound Temple and Young set forth. Being able to compute these solutions seems, after all, to make clear that the nonlinear interactions that come with variations in entropy allow for periodic solutions where otherwise, in the isentropic case, shocks would form.

## 2 The spatial evolution method

To develop the method based on the proof of Temple and Young, we start with the third equation of (1.2) describing conservation of energy,

$$\partial_t\left(\tfrac{1}{2}u^2 + e\right) + \nabla \cdot (up) = 0, \tag{2.1}$$

where $u$ is the velocity, $e$ is the specific internal energy, $p$ is the pressure. The second law of thermodynamics relates the latter two variables, the temperature $\theta$, the specific entropy $s$, and the specifc volume $v$ by

$$de = \theta ds - pdv.$$

With this, we find that (2.1) can be satisfied by

$$\partial_t s = 0. \tag{2.2}$$

In the case of a single spatial dimension, we must not only assume $s = s(x)$ but also $s$ nonconstant. If the entropy is constant, then the problem is isentropic, and we will see that this problem must be nonisentropic to find a periodic solution. So, we choose to consider the case of a piecewise constant entropy profiles.

Our first problem is to find a $T$-periodic solution on the domain

$$x \in [0, l], \qquad t \in [0, T],$$

where $l$ is a chosen finite length and $T$ a parameter to be chosen later. At this time we also choose a background pressure $\bar{p}$, an equation of state $v = v(p, s)$, a $k$-mode to be the dominant term in our solution, an $\alpha \in (0, 1)$ for magnitude of the $k$-mode relative to background pressure, a base frequency $\omega_0$, and an entropy profile $s = s(x)$. Our entropy profile $s(x)$ is taken to be piecewise constant with $L-1$ jumps, which we characterize by $L$ points $x_0, \ldots, x_{L-1}$, where $x_0 = 0$ and $x_{L-1} = l$. With (2.2) satisfied,

$$\begin{cases} \partial_t u + \partial_x p = 0 \\ \partial_t v - \partial_x u = 0 \\ \partial_t\left(\tfrac{1}{2}u^2 + e\right) + \partial_x\left(up\right) = 0 \end{cases}$$

simplifies to

$$\begin{cases} \partial_t u + \partial_x p = 0 \\ \partial_t v - \partial_x u = 0 \end{cases} \tag{2.3}$$

on intervals of constant entropy.

## 2.1 Fréchet derivatives

We take a brief aside at this point to understand what the linearization of a system of differential equations is. We let $X, Y$ be Hilbert spaces, $\Omega$ an open subset of $X$, and $g$ a mapping from $\Omega$ to $Y$. We say that $g$ is *Gâteaux-differentiable* at $u_o \in \Omega$ in the direction of $v \in X$ if the limit

$$d_v g(u_0) \coloneqq \lim_{\delta \to 0} \frac{g(u_0 + \delta v) - g(u_0)}{h}$$

exists, in which case $d_v g(u_0)$ is called the *Gâteaux derivative* of $g$ at $u_0$ in the direction of $v$. In the case of a scalar function $g : \mathbb{R}^n \mapsto \mathbb{R}$ this is equivalent to the usual notion of a directional derivative, and it is in this way that the Gâteaux derivative can be seen as a generalization of the directional derivative, one that can be used in arbitrary, possibly infinite-dimensional as in our case, topological spaces. Of more use for us is the following more restrictive idea of differentiation that generalizes the Jacobian to arbitrary topological spaces. A map $g : X \mapsto Y$ is said to be *Fréchet-differentiable* at $u_0 \in \Omega$ if there exists a bounded linear map $Dg(u_0)$ such that

$$\lim_{||v|| \to 0} \frac{||g(u_0 + v) - g(u_0) - Dg(u_0)[v]||_Y}{||v||_X} = 0$$

or, equivalently,

$$g(u_0 + v) - g(u_0) - Dg(u_0)[v] = o\big(||v||\big).$$

If this is the case, then $Dg(u_0) \in \mathcal{B}(\Omega, Y)$, the best linear approximation of $g$ centered around the point $u_0$, is referred to as the *Fréchet derivative* of $g$ around $u_0$. We now introduce an operator to look at with this notion of a derivative in mind.

Let $\mathcal{E}^{\bar{x}}(\cdot)$ be the spatial evolution operator of the initial boundary value problem consisting of the arbitrary, but in general nonlinear, system of differential equations

$$\partial_t u + \partial_x F(u) = 0, \tag{2.4}$$

together with initial and boundary conditions

$$u(0,t) = u_0(t) \quad \text{and} \quad u \cdot n|_{\partial\Omega} = 0, \quad u(\bar{x}, t) = u(0,t).$$

That is to say, $\mathcal{E}^{\bar{x}}(u_0)$ takes in initial data $u_0(t)$, evolves it nonlinearly from $x = 0$ to $x = \bar{x}$ according to (A.2), and returns $u(\bar{x}, t)$. If we perturb the initial

data of this solution, $u_0(t)$, by some $v_0(t)$ assumed to be such that $(u+v)(x,t)$ is still an exact solution of (A.2), expect with the initial condition

$$(u+v)(0,t) = u_0(t) + v_0(t),$$

then we find that $v(x,t)$ must satisfy

$$\partial_t v + \partial_x \big(F(u+v) - F(u)\big) = 0. \tag{2.5}$$

For a sufficiently differentiable $F$, we can write this as

$$\partial_t v + \partial_x Df(u_0)[v] = o(||v||), \tag{2.6}$$

where $DF(u)$ is the Fréchet derivative of $F(u)$.

Assuming $\mathcal{E}^{\bar{x}}$ is Fréchet-differentiable at $u_0$, then by definition there exists a bounded linear map, which we label $D\mathcal{E}^{\bar{x}}(u_0)$, that satisfies

$$\mathcal{E}^{\bar{x}}(u_0 + U_0) - \mathcal{E}^{\bar{x}}(u_0) - D\mathcal{E}^{\bar{x}}(u_0)[U_0] = o\big(||U_0||\big). \tag{2.7}$$

This map, $D\mathcal{E}^{\bar{x}}(u_0)$, is the evolution operator for the linear approximation of (A.5),

$$\partial_t U + \partial_x Df(u_0)[U] = 0. \tag{2.8}$$

We use uppercase initial data like $U_0(t)$ for initial data that perturbs $u_0(t)$ to a solution satisfying

$$\partial_t(u+U) + \partial_x(u+U) = o(||U||),$$

in contract to lowercase initial data $v_0$ that perturbs $u_0(t)$ to a solution of (A.4), making $u+v$ an exact solution of the same problem. We note that it is $D\mathcal{E}^{\bar{x}}(u_0)[\cdot]$ that is a linear operator, which is represent with brackets. Nonlinear mappings are indicated with parentheses, as in $\mathcal{E}^{\bar{x}}(\cdot)$ or $D\mathcal{E}^{\bar{x}}(\cdot)[U_0]$.

The solution $U$ satisfies

$$\mathcal{E}^{\bar{x}}(u_0 + U_0) - \mathcal{E}^{\bar{x}}(u_0) - U(x_0,t) = o\big(||U_0||\big),$$

and so we write $U = D\mathcal{E}^{\bar{x}}(u_0)[U_0]$ and refer to $U$ as the linearization of $u$ around $u_0$ at, or in the direction of, $U_0$. Both the operator $D\mathcal{E}^{\bar{x}}(u_0)$ and the system of differential equations (A.7) may be referred to as just the linearization, but if necessary some more specificity is required, then we refer to (A.7) as the linearization of the system (A.2). However, as we are solving an initial boundary value problem, it will usually be more useful to talk about the entire linearized problem, consisting of the system (A.7) together with the initial and boundary conditions

$$U(0,t) = U_0(t) \quad \text{and} \quad U \cdot n|_{\partial\Omega} = h(x), \quad U(x,T) = U(x,0).$$

Now, we can return to the problem of solving (2.3).

## 2.2 Linearizing the nonlinear system

We will begin the construction of a periodic solution of (2.3) with an explicit approximation of such a solution. Specifically, we will use a particular linearized version of the problem, and so now derive what exactly the linearized version of the system (2.3) is. Following how (A.7) was arrived at in the previous subsectionm, we assume that we have some exact solution $u, p$ and a perturbation of this, $u + \tilde{u}, p + \tilde{p}$, that is also an exact solution. With these assumptions,

$$\begin{cases} \partial_t \left(u + \tilde{u}\right) + \partial_x \left(p + \tilde{p}\right) = 0 \\ \partial_t v \left(p + \tilde{p}, s\right) - \partial_x \left(u + \tilde{u}\right) = 0. \end{cases}$$

Subtracting the first and second equations of (2.3) from the respective equations above, we find

$$\begin{cases} \partial_t \tilde{u} + \partial_x \tilde{p} = 0 \\ \partial_t \big(v \left(p + \tilde{p}, s\right) - v \left(p, s\right) \big) - \partial_x \tilde{u} = 0. \end{cases}$$

With the fair assumption that the equation of state $v = v(p, s)$ specified at the start of the problem is twice differentiable, we have that

$$v(p + \tilde{p}, s) = v(p, s) + v_p(p, s)\tilde{p} + v_{pp}(\hat{p})\tilde{p}^2,$$

for some $\hat{p}$. Using the notation

$$\sigma^2(p, s) \coloneqq -\frac{\partial v}{\partial p}\Big|_{(p,s)},$$

$\tilde{u}$ and $\tilde{p}$ solve

$$\begin{cases} \partial_t \tilde{u} + \partial_x \tilde{p} = 0 \\ \partial_t \big(\sigma^2(p, s)\tilde{p}\big) + \partial_x \tilde{u} = -v_{pp}(\hat{p})\tilde{p}^2. \end{cases}$$

Again, the linearized system is found by dropping any higher order terms, in this case the $-v_{pp}(\hat{p})\tilde{p}^2$ term in the second equation. Of course, changing the system in this way makes it so that the $U$ and $P$ described by the resulting system

$$\begin{cases} \partial_t U + \partial_x P = 0 \\ \partial_t \left(\sigma^2(p, s)P\right) - \partial_x U = 0, \end{cases} \tag{2.9}$$

do not perturb $u$ and $p$ to another nonlinear solution but rather to an approximation of one.

## 2.3 Solving the linearized problem

This linearization being around an arbitrary $p, u$ does allow for $p, u$ to be a nonconstant state, but we for now we concern ourselves with the linearization around a particular state, one which we will be able to explicitly solve. For whatever our choice of $\bar{p}$, which in our computational examples is $\bar{p} = 101325$ Pa (pascals) as all our parameters are chosen to model air at sea level, we will form our initial approximate solution by perturbing the stationary solution

$$p(x, t) = \bar{p}, \qquad u(x, t) = 0, \qquad s = s(x), \tag{2.10}$$

which is an exact solution of the nonlinear problem, by a solution of the linearized problem around (2.10). That is, we will perturb (2.10) by $P, U$ that solve

$$\begin{cases} \partial_t U + \partial_x P = 0 \\ \bar{\sigma}^2(s)\partial_t P + \partial_x U = 0, \end{cases} \tag{2.11}$$

and satisfy the initial conditions

$$P(0,\cdot) = \text{even}, \qquad U(0,\cdot) = 0, \tag{2.12}$$

boundary conditions

$$d_x P(\cdot,0) = d_x P(\cdot,T) = 0, \qquad U(\cdot,0) = U(\cdot,T) = 0 \tag{2.13}$$

and periodicity conditions

$$\tfrac{\mathcal{I}-\mathcal{R}}{2}\mathcal{S}^{\frac{T}{4}} P(l,\cdot) = 0, \qquad \tfrac{\mathcal{I}-\mathcal{R}}{2}\mathcal{S}^{\frac{T}{4}} U(l,\cdot) = 0. \tag{2.14}$$

We note that in writing (2.11) we are using the notation $\bar{\sigma}^2(s) = \sigma^2(\bar{p}, s)$.

Separating variables in (2.11) by

$$P(x,t) = \phi_j(x)\cos\big(j\omega_0 t\big), \qquad U(x,t) = \psi_j(x)\sin\big(j\omega_0 t\big),$$

where $j \in \mathbb{N}$, we find that $\phi_j$ and $\psi_j$ solve

$$\begin{cases} d_x\phi_j + n\omega_0\psi_j \\ d_x\psi_j - n\omega_0\bar{\sigma}^2\phi_j = 0 \end{cases}$$

on each sub-interval $[x_{i-1}, x_i]$ where $s_i$, and thus $\bar{\sigma}_i$, is constant. We rewrite this as

$$\begin{cases} d_x(\bar{\sigma}\phi_j) + (j\omega_0\bar{\sigma})\psi_j \\ d_x\psi_j - (j\omega_0\bar{\sigma})(\bar{\sigma}\phi_j) = 0, \end{cases}$$

from which we find

$$\frac{d}{dx}\left[R\big(j\omega_0\bar{\sigma}\big)^{-1}\begin{pmatrix} \bar{\sigma}\phi_j \\ \psi_j \end{pmatrix}\right] = 0.$$

So on our sub-intervals of constant entropy, $(\bar{\sigma}_i\phi_j, \psi_j)$ evolves like a rotation,

$$\begin{pmatrix} \bar{\sigma}_i\phi_j(x) \\ \psi_j(x) \end{pmatrix} = R(-j\omega_0\bar{\sigma}_i(x-x_i))\begin{pmatrix} \bar{\sigma}_i\phi_j(x_i) \\ \psi_j(x_i) \end{pmatrix}.$$

We impose continuity of $\phi_j$ and $\psi_j$ at the entropy jumps and so can use the above expression if we just appropriately scale at the start and end of each subinterval. Doing so we find

$$\begin{pmatrix} \phi_j(l) \\ \psi_j(l) \end{pmatrix} = \prod_{i=1}^{L-1} J_{N-i}^{-1}\left(R_{N-i}^{\intercal}\right)^{-j} J_{N-i}\begin{pmatrix} \phi_j(0) \\ \psi_j(0) \end{pmatrix}$$

and, from this,

$$\begin{pmatrix} P(l,t) \\ U(l,t) \end{pmatrix} = \begin{pmatrix} \cos\left(j\frac{2\pi}{T}t\right) & 0 \\ 0 & \sin\left(j\frac{2\pi}{T}t\right) \end{pmatrix} \begin{pmatrix} \phi_j(l) \\ \psi_j(l) \end{pmatrix}, \tag{2.15}$$

where $J_i$ is a scaling matrix to take $\phi_j$ to $\bar{\sigma}_i\phi_j$ and $R_i$ a rotation matrix from $x_i$ to $x_{i+1}$,

$$J_i = \begin{pmatrix} \bar{\sigma}_i & 0 \\ 0 & 1 \end{pmatrix}, \qquad R_i = R\left(\tfrac{2\pi}{T}\bar{\sigma}_i(x_{i+1}-x_i)\right) \qquad \text{for} \qquad i = 1,\ldots,L-1.$$

This is where, as previously referenced, we choose our time period $T$.

If we wish to use our chosen $k$-mode to perturb (2.10), then the $k$-mode must be a solution of entire whole linearized problem. This is to say, the $k$-mode solution

$$P(x,t) = \phi_k(x)\cos(k\omega_0 t) \quad \text{and} \quad U(x,t) = \psi_k(x)\sin(k\omega_0 t) \tag{2.16}$$

must not only solve the linearized system (2.11) but also satisfy the initial, boundary, and periodicity conditions. Since the initial condtions (2.12) and boundary conditions (2.13) are already satisfied by (2.16), it is perhaps unsurprising that it is the periodicity condition (2.14) in particular that will determine our choice of $T$. Specifically, we want to use (2.15) to analyze the condition that both $\frac{\mathcal{I}-\mathcal{R}}{2}\mathcal{S}^{\frac{T}{4}}P(l,\cdot)$ and $\frac{\mathcal{I}-\mathcal{R}}{2}\mathcal{S}^{\frac{T}{4}}U(l,\cdot)$ be zero. Although not how our choice of $T$ will be determined, we introduce here what Temple and Young dub small divisors in [10],

$$\delta_n = \begin{pmatrix} 0 & 1 \end{pmatrix} P^{-n} \prod_{i=1}^{N-1} J_{N-i}^{-1}\left(R_{N-i}^{\intercal}\right)^{-j} J_{N-i} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0. \tag{2.17}$$

In the above, the row vector $\begin{pmatrix} 0 & 1 \end{pmatrix}$ captures the effect of projecting onto the odd component, and $P^{-n}$ captures the effect of shifting by a quarter period with

$$P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We can use this to write the condition that the $k$-mode solve the linearized problem as just $\delta_k = 0$.

To actually find a suitable time period we track the angle of the vector $\begin{pmatrix} P_k(x) & U_k(x) \end{pmatrix}^{\intercal}$ in the $P$-$U$ plane, relative to the $P$-axis, from $x = 0$ to $x = l$. Because of the effect of the quarter period shift, $\delta_k = 0$ is equivalent to the angle reached at $x = l$ being of the form $\frac{(2m-1)\pi}{2}$ or $m\pi$, for even and odd $k$ respectively. The angle at $x = l$ comes from rotating through the first entropy level, rescaling by $\frac{\bar{\sigma}_2}{\bar{\sigma}_1}$, and repeating this for each of the remaining $L-2$ entropy levels. In [10], this is captured by recursively applying the function

$$h(\chi,x) = \begin{cases} \arctan\left(\chi\tan x\right) & -\frac{\pi}{2} < x - k\pi < \frac{\pi}{2} \\ x & x = k\pi \pm \frac{\pi}{2} \end{cases}$$

to the starting angle of zero in the $P$-$U$ plane that our initial data places us at. Denoting the angle by $\gamma$, we see what this looks like in the following algorithm.

**Algorithm 1** Calculating required $T$ for the chosen $k$-mode

1: $\gamma = 0$
2: **for** $n = 1, \ldots, N-2$ **do**
  $\gamma(t) \leftarrow h\big(\frac{\bar{\sigma}_{n+1}}{\bar{\sigma}_n}, k\frac{2\pi}{T}(x_{i+1} - x_i)\big)$
  **end**
3: $\gamma(t) \leftarrow \gamma(t) + k\frac{2\pi}{T}(l - x_{N-1})$
4: $T \leftarrow \texttt{solve}(\gamma = \frac{k\pi}{2})$

In our computations, which were all done within MATLAB®, the value $T$ is set to in line 6 is found with `vpasolve`, found within the Symbolic Math Toolbox. Now we can perturb our stationary solution by the $k$-mode solution $P(0,t) = \cos\left(k\frac{2\pi}{T}t\right)$ to construct our approximate solution to the nonlinear problem that will be refined into a period solution.

## 2.4 Higher linearizations and evolution operators

The nonlinear variables $p$ and $u$ have the same temporal Fourier bases as their linearized counterparts, so

$$p(x,t) = \sum_j p_j(x)\cos(j\omega_0 t), \quad u(x,t) = \sum_j u_j(x)\sin(j\omega_0 t),$$

while $v$ is written as

$$v(x,t) = \sum_j v_j(x)\cos(j\omega_0 t).$$

We write our first approximate solution of the nonlinear problem

$$p^{(0)}(0,t) = p_0^{(0)} = \bar{p} + (\alpha\bar{p})\cos\left(k\omega_0 t\right), \tag{2.18}$$

The superscript denotes this as the first in a series of iteratively improved solutions. Being only an approximation, the residual of (2.18) is nonzero. We set out to further perturb (2.18) by other $j$-modes to improve the residual while fixing the $k$-mode term. That is to say, each step will be of the form

$$p^{(i+1)}(0,t) = p^{(i)}(0,t) + \sum_{j\neq k} r_j \cos\left(j\tfrac{2\pi}{T}t\right).$$

The fact that the $k$-mode component is kept fixed is a key point as this ensures that any solution we arrive at will be nontrivial.

We begin with some explicit corrections to $p^{(0)}(0,t)$,

$$p^{(1)}(0,t) = \bar{p} + (\alpha\bar{p})\cos\left(k\tfrac{2\pi}{T}t\right) + \sum_{j=0,2k} b_j \cos\left(j\tfrac{2\pi}{T}t\right),$$

to improve the accuracy of the solution we start an iterative process from. To do this, and to afterwards design said iterative process, we need to define the map

$$f : p(\cdot, 0) \mapsto \tfrac{\mathcal{I}-\mathcal{R}}{2}\mathcal{S}^{\frac{T}{4}}u(l, \cdot),$$

where $p(0, t)$ is evolved nonlinearly according to (2.3) with $u(0, t) = 0$ and the resulting $u(l, t)$ then shifted by $\frac{T}{4}$ and projected onto its odd component, so

$$f = \frac{\mathcal{I} - \mathcal{R}}{2}\mathcal{S}^{\frac{T}{4}}\mathcal{E}^l,$$

where $\mathcal{E}$ is essentially the evolution operator seen in the earlier part of this section on Fréchet derivatives. With this addition of the shift and projection to the evolution operator, solutions of the nonlinear problem can be characterized by $f = 0$. We take the Fréchet derivative of $f$ around the state (2.10) to find the corresponding operator for the linearized problem,

$$Df(\bar{p}) : P(0, \cdot) \mapsto \tfrac{\mathcal{I}-\mathcal{R}}{2}\mathcal{S}^{\frac{T}{4}}U(l, \cdot),$$

which evolves the initial data $P(0, t)$ according to (3.4) rather than the nonlinear system. Similarly, we can write this with the Fréchet derivative of the evolution operator,

$$Df = \frac{\mathcal{I} - \mathcal{R}}{2}\mathcal{S}^{\frac{T}{4}}D\mathcal{E}^l(\bar{p}).$$

We only have need to take one more Fréchet derivative in this way, which gives the bilinear operator which is the bilinear operator

$$D^2 f(p) : P^{(1)}(0, \cdot) \times P^{(2)}(0, \cdot) \mapsto \frac{\mathcal{I} - \mathcal{R}}{2}\mathcal{S}^{\frac{T}{4}}U^{[1,2]}(l, \cdot).$$

In the above, $U^{[1,2]}(l, \cdot)$ is found by evolving the initial data

$$P^{[1,2]}(0, \cdot) = 0, \qquad U^{[1,2]}(0, \cdot) = 0$$

according to the linearization of (2.11),

$$\begin{cases} \partial_t U^{[1,2]} + \partial_x P^{[1,2]} = 0 \\ \bar{\sigma}^2 \partial_t P^{[1,2]} + \partial_x U^{[1,2]} = -v_{pp}(\bar{p}, s)\partial_t\big(P^{(1)}P^{(2)}\big). \end{cases}$$

Here, $P^{(1)}$ is simultaneously evolving according to

$$\begin{cases} \partial_t U^{(1)} + \partial_x P^{(1)} = 0 \\ \bar{\sigma}^2(s) \partial_t P^{(1)} + \partial_x U^{(1)} = 0, \end{cases}$$

and $P^2(2)$ is simultaneously evolving according to an analogous system to the above.

Each step of improving the solution will be of the form

$$p^{(i+1)}(0, t) = p^{(i)}(0, t) - \sum_{j \neq k} r_j \cos\big(j\tfrac{2\pi}{T}t\big).$$

Expanding $f$ around some $p^{(i+1)}_{(0,t)}$, we write

$$\begin{aligned} f\big(p^{(i+1)}_{(0,t)}\big) &= f\Big(p^{(i)}_{(0,t)} + \sum_{j\neq k} r^{(i)}_j \cos\big(j\tfrac{2\pi}{T}t\big)\Big) \\ &= f\big(p^{(i)}_{(0,t)}\big) + \sum_{j\neq k} r^{(i)}_j Df\big(p^{(i)}_{(0,t)}\big)\big[\cos\big(j\tfrac{2\pi}{T}t\big)\big] + \mathcal{O}(\alpha^4) \\ &= f\big(p^{(i)}_{(0,t)}\big) + \sum_{j\neq k} r^{(i)}_j Df\big(\bar{p}\big)\big[\cos\big(j\tfrac{2\pi}{T}t\big)\big] \\ &\quad + \sum_{j\neq k} \alpha r^{(i)}_j D^2 f(\bar{p})\big[\cos\big(j\tfrac{2\pi}{T}t\big), \tfrac{1}{\bar{\sigma}}\phi_j\big] + \mathcal{O}(\alpha^4). \end{aligned} \tag{2.19}$$

We can design many different iteration methods in this way. This dissertation covers two different quasi-Newton methods, the more accurate but slower

$$p^{(i+1)}_{(0,t)} = p^{(i)}_{(0,t)} - \Big(Df(\bar{p}) + (\alpha\bar{p})D^2 f(\bar{p})\big[\cos\big(k\tfrac{2\pi}{T}t\big)\big]\Big)^{-1}\Big[f\big(p^{(i)}_{(0,t)}\big)\Big] \tag{2.20}$$

and the faster but less accurate

$$p^{(i+1)}_{(0,t)} = p^{(i)}_{(0,t)} - Df(\bar{p})^{-1}\Big[f\big(p^{(i)}_{(0,t)}\big)\Big]. \tag{2.21}$$

To actually implement either of these methods though, we need a way to find $f$, $Df$, and $D^2 f$ and an appropriate way to represent these as well as $p^{(i+1)}_{(0,t)}$.

## 2.5 Iteration for the nonlinear problem

Our variables have infinite-dimensional Fourier bases, so, firstly, we take care of the issue that comes with all infinite-dimensional problems, namely their ceaseless hunger for a computer's resources. We limit the bases for $p$, $v$, and $u$ to finite subsets of $N$ eigenfunctions and write these variables as vectors on those subsets,

$$p^{(i)}(x_0,t) = \begin{pmatrix} p^{(i)}_0(x_0) \\ \cdots \\ p^{(i)}_{k-1}(x_0) \\ p^{(i)}_{k+1}(x_0) \\ \cdots \\ p^{(i)}_N(x_0) \end{pmatrix}, \quad v^{(i)}(x_0,t) = \begin{pmatrix} v^{(i)}_1(x_0) \\ \cdots \\ v^{(i)}_{k-1}(x_0) \\ v^{(i)}_k(x_0) \\ \cdots \\ v^{(i)}_N(x_0) \end{pmatrix}, \quad u^{(i)}(x_0,t) = \begin{pmatrix} u^{(i)}_1(x_0) \\ \cdots \\ u^{(i)}_{k-1}(x_0) \\ u^{(i)}_k(x_0) \\ \cdots \\ u^{(i)}_N(x_0) \end{pmatrix}.$$

For $p$, we omit the $k$-mode term since it is fixed. Similarly, $v$ and $u$ are written as vectors over a subset missing the 0-mode as $v_0$ and $u_0$ are fixed. The operator $Df(\check{p})$, as a map from the $p$-space to the $u$-space, can then represented as a matrix mapping $p$-basis vectors to $u$-basis vectors,

$$Df(\check{p}) = \begin{pmatrix} d_1 & \cdots & d_{k-1} & d_{k+1} & \cdots & d_N \end{pmatrix} \quad \text{where} \quad d_n = Df(\check{p})\big[\cos\big(n\tfrac{2\pi}{T}t\big)\big]$$

That is, the $j$th column of $Df(\check{p})$ as a matrix is the $u$-basis vector the $j$th $p$-basis vector is mapped to,

$$Df(\check{p})\big[\cos(n\tfrac{2\pi}{T}t)\big] = \begin{pmatrix} \big\langle Df(\check{p})\big[\cos\left(n\tfrac{2\pi}{T}t\right)\big];\psi_1\big\rangle \\ \cdots \\ \big\langle Df(\check{p})\big[\cos\left(n\tfrac{2\pi}{T}t\right)\big];\psi_N\big\rangle \end{pmatrix} = \begin{pmatrix} \nu_{n,1} \\ \cdots \\ \nu_{n,N} \end{pmatrix}.$$

This means $Df(\check{p})$ can be written as the $N \times N$ matrix

$$Df(\check{p}) = \begin{pmatrix} \nu_{0,1} & \cdots & \nu_{k-1,1} & \nu_{k+1,1} & \cdots & \nu_{N,1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \nu_{0,k} & \cdots & \nu_{k-1,k} & \nu_{k+1,k} & \cdots & \nu_{N,k} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \nu_{0,N} & \cdots & \nu_{k-1,N} & \nu_{k+1,N} & \cdots & \nu_{N,N} \end{pmatrix}.$$

We know $Df(\bar{p})$ respects modes,

$$\Big\langle Df(\bar{p})\big[\cos\left(n\tfrac{2\pi}{T}t\right)\big], \sin\left(m\tfrac{2\pi}{T}t\right)\Big\rangle \neq 0 \quad \text{only if} \quad n = m,$$

so except for the $k$th row, the only nonzero elements are the $\nu_{n,n}$ terms, given by (2.17). Looking at (2.20) as $N$ equations, this means in (2.20) we solve the $n$-mode equation with the $n$-mode variable, except for $n = k$. This means

$$r_n^{(i)} = \frac{1}{\delta_n}\Big\langle f\big(p_{(0,t)}^{(i)}\big); \sin\left(n\tfrac{2\pi}{T}t\right)\Big\rangle = \frac{1}{\delta_n}u_n^{(i)}(l) \quad \text{for} \quad n = 1,\ldots,N,\ n \neq k.$$

We are left with a $k$-mode equation that cannot be solved like this with no $k$-mode variable. However there is no 0-mode equation, so the 0-mode variable is unused, which Temple and Young use to solve the $k$-mode equation by turning it into a bifurcation problem. In this way, we use the leading order effect of the 0-mode and $k$-mode interacting, and so define

$$\delta_0 = -(\alpha\bar{p})D^2 f(\bar{p})\big[1, \cos\left(k\tfrac{2\pi}{T}t\right)\big], \tag{2.22}$$

which can be explicitly calculated with Duhamel's principle. Now,

$$Df(\bar{p}) = \begin{pmatrix} 0 & \delta_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \delta_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \delta_{k-1} & 0 & \cdots & 0 \\ \delta_0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \delta_{k+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \delta_N \end{pmatrix}$$

We can find $D^2 f(\bar{p})\big[\cos\left(k\tfrac{2\pi}{T}t\right)\big]$ as a matrix in a similar way. It will not be analogous to a diagonal matrix as $Df(\bar{p})$ is, but we can find all the entries by

explicitly calculating what looks like some kind of second order small divisor, the interaction of the $k$-mode each $n$-mode variable,

$$\delta_{n,m} = \Big\langle D^2 f(\bar{p})\big[\cos\big(k\tfrac{2\pi}{T}t\big), \cos\big(n\tfrac{2\pi}{T}t\big)\big], \sin\big(m\tfrac{2\pi}{T}t\big)\Big\rangle,$$

which can be done with Duhamel's principle. If we wish to use (2.21), we can discard the $\delta_0$ in the $k$th row of $Df(\bar{p})$, as the formula above for $n = 0, m = k$ captures exactly the effect $\delta_0$ accounts for.

Here we point out the reason for requiring that $s$ be nonconstant. In the case of $Df(\bar{p})$, if $s$ is constant then some modes that are resonant with the $k$-mode,

$$Df(\bar{p})\big[\cos(j\tfrac{2\pi}{T}t)\big] = 0 \quad \text{for} \quad j = mk,\ m \in \mathbb{N}^+,$$

as for every $m \in \mathbb{N}$ we would find a $j \in \mathbb{N}$ such that $\lambda_j = m\lambda_k$. So we require $s$ nonconstant, making the problem nonresonant and thereby avoiding any issues with the use of $Df(\bar{p})^{-1}$ in (2.20) or (2.21). What remains is finding $f\big(p^{(i)}_{(0,t)}\big)$, $Df(\bar{p})$, and $D^2 f(\bar{p})\big[\cos\big(k\tfrac{2\pi}{T}t\big)\big]$ as matrices. Once done, iterating with any quasi-Newton method is just a matter of solving a matrix equation.

## 2.6 Mixed finite-difference spectral methods

While we are able to derive a formula for two of these terms, $Df(\bar{p})$, and $D^2 f(\bar{p})\big[\cos(k\tfrac{2\pi}{T}t)\big]$, we must numerically calculate $f\big(p^{(i)}_{(0,t)}\big)$. This is no surprise as $f$ is a highly nonlinear operator, so, as the title of this subsection might suggest, we use a mixed finite-difference spectral method. Specifically, we use this kind of method for finding $f\big(p^{(i)}_{(0,t)}\big)$. To start, we project the first equation in (2.3) onto $\cos(n\frac{2\pi}{T}t)$ and the second onto $\sin(n\frac{2\pi}{T}t)$,

$$\begin{cases} d_x p_n + n\frac{2\pi}{T} u_n = 0 \\ d_x u_n + n\frac{2\pi}{T} v_n = 0. \end{cases}$$

Because we have Fourier bases, the fast Fourier transform (FFT) is available for decomposing our variables here. We use this system to design the finite-difference part of this scheme, choosing the leap-frog type method

$$\begin{cases} p_n^{j+1} = p_n^{j-1} - \big(2n\frac{2\pi}{T}\Delta x\big)\, u_n^j \\ u_n^{j+1} = u_n^{j-1} - \big(2n\frac{2\pi}{T}\Delta x\big)\, v_n^j. \end{cases} \tag{2.23}$$

At $x = l$, we will know what $u$ is as a linear combination of $\psi_j$'s, so we can also find the combo of $\psi_j$'s after shifting $\frac{T}{4}$ and projecting onto the odd component.

To summarize the process of using the above system, our initial data $p^{(0)}_{(0,t)}$ (2.18) and the then calculated $v^{(0)}_{(0,t)} = v\big(p^{(0)}_{(0,t)}, s(0)\big)$ are decomposed using the FFT. Since $u_j^0 = 0$ for all $j$, we know every $p_j^{(0)}(0), v_j^{(0)}(0), u_j^{(0)}(0)$. The spatial domain is discretized so that we have some chosen number $M \in \mathbb{N}$ of

evenly space points. That is, for the interval $[0, l]$ we use the discretization $x_1, \ldots, x_M$ where $x_{i+1} - x_i = \Delta x = \frac{l}{M}$. We extend the initial data to $x_2$ and find each $p_j^{(0)}(\Delta x), v_j^{(0)}(\Delta x), u_j^{(0)}(\Delta x)$ and now apply (2.23). Doing so gives each $v_j^{(0)}(2\Delta x)$ and $u_j^{(0)}(2\Delta x)$. We recompose $v^{(0)}(t, 2\Delta x)$ and use the inverted equation of state $p = p(v, s)$ to find $p^{(0)}(t, 2\Delta x)$. After decomposing for each $p_j^{(0)}(2\Delta x)$, we can again apply (2.20), continuing on in this fashion until we find each $p_j^{(0)}(M\Delta x), v_j^{(0)}(M\Delta x), u_j^{(0)}(M\Delta x)$.

After shifting and projecting, we finally have $\frac{\mathcal{I}-\mathcal{R}}{2} S^{\frac{T}{4}} u^{(0)}(t, l) = f(p^{(0)}_{(0,t)})$ as a $u$-basis vector. The remaining terms in (2.20), $Df(\bar{p})$ and $D^2 f(\bar{p})\big[\cos(k\frac{2\pi}{T}t)\big]$, are linear maps from the $p$-space to the $u$-space, and so can be written as matrices if we just find the $u$-basis vector image of each $p$-basis element $\cos(j\frac{2\pi}{T}t)$ as a vector.

**Algorithm 2** Nonlinear one-dimensional spatial evolution

We start the problem process knowing the entropy profile $s(x)$, equation of state $v = v(p, s)$, choice of $k$-mode, time period $T$, and the starting approximate solution $p^{(0)}(x, 0)$.

1: xGrid $= [0 : \Delta x : l]$
2: tGrid $= [0 : \Delta t : T]$
3: $p(0, 0, :) = \bar{p} + (\alpha\bar{p}) \cos(k\frac{2\pi}{T} xGrid)$
4:
5: $u(0, 0, :) = 0$
6: $v(0, 0, :) = v\big(p(0, 0, :), s(:)\big)$

## 3 The temporal evolution method

The next main focus of this dissertation is the development of a method of computing periodic solutions of the compressible Euler equations of an overall similar structure but with the key difference in the use of evolution through time where we previously used evolution through space. The advantage of this approach is that we can start with a more arbitrary spatial domain, which in particular no longer need be one-dimensional. That is to say, the spacetime domain is now represented by

$$x \in \Omega \qquad \text{and} \qquad t \in [0, 2T],$$

where $\Omega$ is a spatial domain which can be chosen to be of any desired dimensionality. However, one should be forewarned of the analytical and computational difficulties that come along with $\Omega$'s of high dimensionality.

We also assume some choice of a number of necessary parameters, namely a background pressure $\bar{p}$, a particular $k$-mode eigenfunction to use to perturb the background pressure, an $\alpha \in (0, 1)$ with which we set the magnitude of the perturbation to be $\alpha\bar{p}$, an equation of state $v = v(p, s)$, and an entropy profile

$s = s(x)$. In alignment with the previously referenced change to using time evolution in place of space evolution, the problem is now to look for $2T$-periodic solutions of the Euler equations, which to account for higher possible spatial dimensions we now write as

$$\begin{cases} \partial_t u + \nabla p = 0 \\ \partial_t(-v) + \nabla \cdot u = 0 \\ \partial_t(\frac{1}{2}u^2 + e) + \nabla \cdot (up) = 0 \end{cases}$$

or factoring in that $s = s(x)$ as

$$\begin{cases} \partial_t u + \nabla p = 0 \\ \partial_t(-v) + \nabla \cdot u = 0, \end{cases} \tag{3.1}$$

with initial and boundary conditions

$$u(x,0) = 0 \qquad \text{and} \qquad u \cdot n\Big|_{\partial\Omega} = \nabla p \cdot n\Big|_{\partial\Omega} = 0.$$

Here, $T$ is a parameter that itself is chosen by our choice of $k$-mode. How a particular $k$-mode determines a choice to make for $T$ we will see shortly. Just as this version of the problem prescribes an initial condition $u(x,0) = 0$ in place of the previous $u(0,t) = 0$, we can frame this problem as looking for an initial pressure profile $p(x,0)$ to satisfy a periodicity condition at a *later point in time*,

$$p(x,2T) = p(x,0), \qquad u(x,2T) = u(x,0) = 0,$$

in place of the previous framing of looking for a $p(0,t) = p_0(t)$ to satisfy a periodicity condition at a *farther point in space*. To reduce computational costs later on though, we can calculate the nonlinear evolution from $t = 0$ to $t = T$ and check the equivalent conditions

$$p(x,T) = -p(x,0), \qquad u(x,T) = -u(x,0) = 0, \tag{3.2}$$

saving the computational cost of evolving all the way out to $t = 2T$.

## 3.1 Solving the linearized problem

For the time evolution method, a starting approximate solution is constructed in an analogous way to the previous construction (2.18), namely by taking the stationary solution

$$p(x,t) = \bar{p}, \qquad u(x,t) = 0, \tag{3.3}$$

and perturbing it by a chosen $k$-mode solution of the linearized problem. So, linearizing (3.1) around (3.3), we find the linearized problem to consist of the system

$$\begin{cases} \partial_t U + \nabla P = 0 \\ \bar{\sigma}^2 \partial_t P + \nabla \cdot U = 0, \end{cases} \tag{3.4}$$

which is the the multidimensional version of (2.11), the initial and boundary conditions

$$U(x,0) = 0 \qquad \text{and} \qquad U \cdot n\Big|_{\partial\Omega} = \nabla P \cdot n\Big|_{\partial\Omega} = 0,$$

and the $2T$-periodicity condition

$$P(x,2T) = P(x,0), \qquad U(x,2T) = U(x,0) = 0,$$

which, as we pointed out for the nonlinear problem, allows us to check the the equivalent conditions

$$P(x,T) = -P(x,0), \qquad U(x,T) = -U(x,0) = 0.$$

We separate variables with the assumption that $P$ and $U$ are of the form

$$P(x,t) = a(t)\frac{\phi(x)}{\bar{\sigma}(s)}, \qquad U(x,t) = b(t)\psi(x).$$

Plugging the above into (3.4), we find

$$\begin{cases} b'(t)\psi(x) + a(t)\nabla\big(\frac{\phi(x)}{\bar{\sigma}(s)}\big) = 0 \\ \bar{\sigma}^2(s)a'(t)\frac{\phi(x)}{\bar{\sigma}(s)} + b(t)\nabla\cdot\psi(x) = 0. \end{cases}$$

Making the choice that

$$b'(t) = -\lambda a(t), \qquad a'(t) = \lambda b(t),$$

we see that $\frac{1}{\bar{\sigma}}\phi_n, \psi_n, \lambda_n$ are eigenfunction-eigenvalue solutions of

$$\begin{cases} -\nabla\big(\frac{1}{\bar{\sigma}}\phi_j\big) = \lambda_j\psi_j \\ \nabla\cdot\psi_j = \bar{\sigma}^2\lambda_j\big(\frac{1}{\bar{\sigma}}\phi_j\big) \end{cases} \quad \text{with} \quad \psi\cdot\hat{n} = 0, \quad \nabla\phi\cdot\hat{n} = 0. \tag{3.5}$$

However, solving (3.5), or the equivalent problem

$$-\Delta\big(\tfrac{1}{\bar{\sigma}}\phi_j\big) = \bar{\sigma}^2\lambda_j^2\big(\tfrac{1}{\bar{\sigma}}\phi_j\big),$$

requires that the spatial domain $\Omega$ and the entropy profile $s(x)$ on said domain both be specified, so all that we can really do at this point note the case of $j = 0$ we will have $\lambda_0 = 0$, $\frac{\phi_0}{\bar{\sigma}} = c$ for $c$ some constant, and $\psi_0 = 0$ for whatever the appropriate zero vector is for a given $\Omega$. In the case of a finite one-dimensional spatial domain this is a Sturm-Liouville problem. Although it may take slightly more care to show this in higher spatial dimensions, in the 1-D case classical Sturm-Liouville theory tells us that for $\Omega = [0,l]$ the set of eigenfunctions $\{\frac{\phi_n}{\bar{\sigma}}\}$ forms an orthonormal basis of $L^2_{w_1}([0,l])$, where $w_1$ specifies the weight of the inner product. In the case of (3.5), the eigenfunctions $\frac{\phi_n}{\bar{\sigma}}$ are orthogonal when the inner product is weighted by $w_1(x) = \bar{\sigma}^2$.

For future use, we note that this implies also that $\{\phi_n\}$ is an orthonormal basis of $L^2([a,b])$ with the usual inner product and $\{\bar{\sigma}\phi_n\}$ an orthonormal basis of $L^2_{w_2}([a,b])$ where $w_2(x) = \bar{\sigma}^{-2}$. Additionally, using

$$\psi_j = \frac{1}{\bar{\sigma}} \nabla \Big( \frac{\phi_j}{\bar{\sigma}} \Big)$$

we find that $\{\psi_j\}$ is an orthonormal basis of $L^2([a,b])$ with the usual inner product. Thus, if $P$ and $U$ are decomposed in terms of $\{\frac{\phi_j}{\bar{\sigma}}\}$ and $\{\psi_j\}$ respectively,

$$P(x,t) = \sum_j P_j(t) \frac{\phi_j(x)}{\bar{\sigma}(s)}, \qquad U(x,t) = \sum_j U_j(t)\psi_j(x),$$

then any particular component $P_j$ or $U_j$ can be found by the inner product $\big\langle P, \frac{\phi_j}{\bar{\sigma}} \big\rangle$ or $\langle U; \psi_j \rangle$, provided we use the correct weighted inner product. For clarity, inner products will have a subscript $w_k$ after the right bracket to indicate the specific weighted inner product,

$$\langle f, g \rangle_w = \int_\Omega f(x) \cdot \overline{g(x)} \cdot w(x)\, dx.$$

Since $\{\frac{\phi_j}{\bar{\sigma}}\}$ is the basis for $P$, we decompose $P$ with the $w_1(x) = \bar{\sigma}^2(x)$ weighted inner product,

$$P_j = \Big\langle P, \frac{\phi_j}{\bar{\sigma}} \Big\rangle_{w_1} = \int_\Omega P \frac{\phi_j}{\bar{\sigma}}\, w_1(x)\, dx.$$

And with $\{\psi_j\}$ as the basis for $U$, we use the usual inner product to find a component of $U$,

$$U_j = \langle U; \psi_j \rangle_{L^2} = \int_\Omega U \cdot \psi_j \, dx.$$

With this, we can now continue on with solving the linearization.

## 3.2 Independent $j$-mode systems.

For now, we will assume that we have solved the eigenvalue problem (3.5) for the eigenvalue-eigenfunction solutions $\phi_j(x), \psi_j(x), \lambda_j(x)$. How to solve this eigenvalue problem can vary greatly, depending on the chosen spatial domain $\Omega$ and entropy profile $s(x)$, as we will later see with different domains and entropy profiles in our computational examples. But the process with which the eigenfunctions are used to construct a periodic solution, is always the same.

To continue on with solving the linearization (3.4), we take the inner product of each equation with an eigenfunction from the corresponding basis of the $\partial_t$ term. The term $\partial_t U$ has basis $\{\psi_j\}$, the same basis as for $U$ since differentiating in time does not affect the eigenfunction basis. Thus, we take the usual inner product of the first equation in (3.4) and $\psi_j$. Similarly, the term $\bar{\sigma}^2(s)\partial_t P$ has

basis $\{\bar{\sigma}\phi_j\}$, meaning we take the weighted inner product of the second equation in (3.4) and $\bar{\sigma}\phi_j$, with weight $w_2 = \bar{\sigma}^{-2}$. Taking these inner products thus yields

$$\begin{cases} \langle \partial_t U; \psi_j \rangle_{L^2} + \langle \nabla P; \psi_j \rangle_{L^2} = 0 \\ \langle \bar{\sigma}^2(s)\partial_t P, \bar{\sigma}\phi_j \rangle_{w_2} + \langle \nabla \cdot U, \bar{\sigma}\phi_k \rangle_{w_2} = 0. \end{cases} \tag{3.6}$$

We look at these two equations one at a time. For the first equation, moving the time derivative outside the first term and integrating the second term by parts gives

$$d_t \langle U; \psi_j \rangle_{L^2} - \langle P, \nabla \cdot \psi_j \rangle_{L^2} + \int_{\partial\Omega} P\psi_j \cdot \hat{n}\, ds = 0.$$

The first term is now exactly the $\psi_j$ component of $U$, and with the boundary conditions in (3.4), the integral along $\partial\Omega$ disappears. Using the eigenvalue problem (3.5) to rewrite the middle term, the whole equation becomes

$$d_t U_j - \lambda_j \langle P, \bar{\sigma}\phi_j \rangle_{L^2} = 0. \tag{3.7}$$

We note that in the second term of the above, $\langle P, \bar{\sigma}\phi_j \rangle_{L^2}$ cannot immediately be written in terms of the component $P_j$. Firstly, $P$ is expressed in the basis $\{\frac{\phi_j}{\bar{\sigma}}\}$, and $\{\bar{\sigma}\phi_j\}$ is not even an orthogonal basis with the usual inner product. However, knowing that $\{\frac{\phi_j}{\bar{\sigma}}\}$ is an orthogonal basis with an inner product weighted by $w_1(x) = \bar{\sigma}^2$, we can manipulate (3.7) so that the component $P_j$ appears,

$$\langle P, \bar{\sigma}\phi_j \rangle_{L^2} = \int_\Omega P\bar{\sigma}\phi_j \, dx = \int_\Omega P\frac{\phi_j}{\bar{\sigma}}\, \bar{\sigma}^2 \, dx = \left\langle P, \frac{\phi_j}{\bar{\sigma}} \right\rangle_{w_1} = P_j.$$

With this the first equation in (3.6) finally becomes

$$d_t U_j - \lambda_j P_j = 0.$$

We handle the second equation similarly. Again moving the $\partial_t$ outside of the first term to write $d_t\langle \bar{\sigma}^2 P, \bar{\sigma}\phi_j \rangle_{w_2}$, we further manipulate this term to find

$$\begin{aligned} d_t\langle \bar{\sigma}^2 P, \bar{\sigma}\phi_j \rangle_{w_2} &= d_t \int_\Omega \left(\bar{\sigma}^2 P\right)\left(\bar{\sigma}\phi_j\right)\, \frac{1}{\bar{\sigma}^2}\, dx = d_t \int_\Omega P\Big(\frac{\phi_j}{\bar{\sigma}}\Big)\, \bar{\sigma}^2 \, dx \\ &= d_t\Big\langle P, \frac{\phi_j}{\bar{\sigma}} \Big\rangle_{w_1} = d_t P_j. \end{aligned}$$

We would like to again integrate the second term by parts and use (3.5) to get some inner product involving $U$ and $\psi_j$. However, doing so immediately yields $\langle U; \nabla\left(\bar{\sigma}\phi_j\right)\rangle_{w_2} - \int_{\partial\Omega} \bar{\sigma}\phi_j U \cdot \hat{n} s$, which the eigenvalue problem cannot help with. Instead, we first change the inner product weight and then integrate by parts,

$$\begin{aligned} \langle \nabla \cdot U, \bar{\sigma}\phi_j \rangle_{w_2} &= \int_\Omega \left(\nabla \cdot U\right)\left(\bar{\sigma}\phi_j\right)\, \frac{1}{\bar{\sigma}^2}\, dx = \int_\Omega \left(\nabla \cdot U\right)\Big(\frac{\phi_j}{\bar{\sigma}}\Big)\, dx \\ &= -\int_\Omega U \cdot \nabla\Big(\frac{\phi_j}{\bar{\sigma}}\Big)\, dx + \int_{\partial\Omega} \Big(\frac{\phi_j}{\bar{\sigma}}\Big) U \cdot \hat{n}\, ds. \end{aligned}$$

Now, continuing from this point, we can again use (3.5), in particular the relation $\lambda_j \psi_j = -\nabla\big(\frac{\phi_j}{\bar{\sigma}}\big)$, along with the boundary conditions in (3.4) to write

$$\begin{aligned}\langle \nabla \cdot U, \bar{\sigma}\phi_j \rangle_{w_2} &= \cdots \\ &= -\int_\Omega U \cdot \nabla\Big(\frac{\phi_j}{\bar{\sigma}}\Big)\, dx = \lambda_j \int_\Omega U \cdot \psi_j \, dx \\ &= \lambda_j \langle U; \psi_j \rangle_{L^2}.\end{aligned}$$

Recognizing $\langle U; \psi_j \rangle_{L^2}$ as the component $U_j$, the second equation is then

$$d_t P_j + \lambda_j U_j = 0.$$

So, for whatever the eigenvalue-eigenfunction solutions $\phi_j, \psi_j, \lambda$ of (3.5) may be, they can be used to turn the linearization (3.4) into independent systems for each of the component pairs $P_j, U_j$,

$$\begin{cases} d_t P_j + \lambda_j U_j = 0 \\ d_t U_j - \lambda_j P_j = 0. \end{cases} \tag{3.8}$$

The components $P_j, U_j$ at any given time thus depend only on the previous values of $P_j$ and $U_j$. This is to say, the linearization around the quiet state respects modes, which is inline with our expectation that the linearization around a constant state lack any of the nonlinear behavior that may be seen in the original system.

## 3.3 Choosing $T$ to solve the linearization.

We now look to solve an arbitrary one of the previously derived $k$-mode systems. Each pair of components $P_j, U_j$ only depend on their own previous values, so if we know the decompositions of $P$ and $U$ at $t = 0$, then each (3.8) tells us what those components are at any future time. An arbitrary (3.8) becomes the matrix equation

$$d_t \begin{pmatrix} P_j \\ U_j \end{pmatrix} + \lambda_j \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} P_j \\ U_j \end{pmatrix} = 0,$$

which is rewritten further as

$$d_t \left[ R(-\lambda_j t) \begin{pmatrix} P_j \\ U_j \end{pmatrix} \right] = 0.$$

Thus the values of the $k$-components $P_j(t), U_j(t)$ depend only on their own initial values $P_j(0), U_j(0)$ according to

$$\begin{pmatrix} P_j \\ U_j \end{pmatrix}_t = R(\lambda_j t) \begin{pmatrix} P_j(0) \\ U_j(0) \end{pmatrix}. \tag{3.9}$$

We can now construct a solution of the linearization (3.4) with our previously chosen $k$-mode.

Again, the linearized problem asks us to find initial data $P(x,0) = P_0(x)$ such that when $P_0(x)$ and $U(x,0) = 0$ are evolved according to the linearization (3.4), we find $U(x,T) = 0$. If we take initial data

$$P(x,0) = \beta \frac{\phi_k(x)}{\bar{\sigma}(s)}, \qquad U(x,0) = 0,$$

then by (3.9),

$$P_k(t) = \beta \cos(\lambda_k t), \qquad U_k(t) = \beta \sin(\lambda_k t). \tag{3.10}$$

We refer to the resulting coefficients $U_k(T)$ as small divisors and denote them by $\delta_k$. So,

$$\delta_k = \sin(\lambda_k T)$$

So if we impose the choice of time period

$$T = \frac{\pi}{\lambda_k}, \tag{3.11}$$

then $\delta_k = 0$ and (3.10) is in fact a $2T$-periodic solution of the linearization. We still have the task of actually finding the eigenvalue-eigenfunction solutions $\phi_j, \psi_j, \lambda_j$ of (3.5) for a given $\Omega$ and $s(x)$. But, more importantly, once we have those eigenfunctions and eigenvalues, we have a way to construct solutions of the linearization (3.4) of the form (3.10) by just picking the right time period. Although such solutions are not going to be exact solutions of the nonlinear problem, these solutions of the linearization are useful nonetheless in that they can serve as rough approximations of a solution.

We again require that the problem be nonresonant, which means ensuring that

$$\delta_j = \sin(\pi \tfrac{\lambda_j}{\lambda_k}) \neq 0 \quad \text{for all} \quad j \neq k \tag{3.12}$$

as resonance greatly complicate matters and is outside the scope of this dissertation. However, this requirement looks to have a different appearance on different domains. If we are working on a one-dimensional spatial domain like $\Omega = [a,b]$, then as in the last section, we again need $s$ nonconstant for the problem to be nonresonant and so choose for $s$ to be piecewise constant. Based on the calculations done in the course of this dissertation, this seems to not be a concern for higher dimensional spatial domains. The numerous computations with a constant entropy profile on the disc, the annulus, and the ball, these being the primary higher spatial dimension domains considered in this dissertation, showed the nonresonance condition given by (3.12) to still be satisfied. And so in these cases, the simpler $s = s_0$ will be used instead of just the more complicated piecewise constant $s = s(x)$ used in the previous section.

## 3.4 The nonlinear components $u_k, p_k, v_k$.

We now turn our attention back to the original nonlinear system. We separate variables in (3.1) with the assumption that $p$ and $u$ are expressed in the same

eigenfunction bases as their linearized counterparts $P$ and $U$, which is to say in the bases $\{\frac{\phi_j}{\bar{\sigma}}\}$ and $\{\psi_j\}$, respectively. So,

$$p(x,t) = \sum_j p_j(t)\frac{\phi_j(x)}{\bar{\sigma}(s)}, \qquad u(x,t) = \sum_j u_j(t)\psi_j(x).$$

We also need a basis for $v$, but $v$ does not have a linearized counterpart to look to for this. Instead, we go to the second equation of the p-system, $\partial_t v - \nabla\cdot u = 0$, to note that $v$ should be expressed in the same basis as $\nabla\cdot u$. With the eigenvalue problem (3.5),

$$-\partial_t v = \nabla\cdot u = \sum_j u_j(t)\left(\nabla\cdot\psi_j(x)\right) = \sum_j \lambda_j u_j(t)\bar{\sigma}(s)\phi_j(x),$$

so $v$ has its own eigenfunction basis, $\{\bar{\sigma}\phi\}$, and is written

$$v(x,t) = \sum_j v_j(t)\bar{\sigma}(s)\phi_j(x).$$

We saw earlier that this basis is orthogonal with the inner product weighted by $w_2 = \bar{\sigma}^{-2}$. Any component can then be found by

$$u_j = \langle u;\psi_j\rangle_{L^2}, \quad p_j = \left\langle p, \frac{\phi_j}{\bar{\sigma}}\right\rangle_{w_1}, \quad v_j = \langle v, \bar{\sigma}\phi_j\rangle_{w_2}.$$

We can attempt to proceed as with the linearization, taking the inner product of the two equations in the nonlinear system with an element from the bases of the first terms in each equation,

$$\begin{cases} \langle\partial_t u;\psi_j\rangle_{L^2} + \langle\nabla p;\psi_j\rangle_{L^2} = 0 \\ \langle\partial_t(-v),\bar{\sigma}\phi_j\rangle_{w_2} + \langle\nabla\cdot u,\bar{\sigma}\phi_j\rangle_{w_2} = 0. \end{cases} \tag{3.13}$$

Trying to manipulate these equations in a similar fashion to (3.6), we start with the first equation, moving the time derivative outside of the first term and integrating the second by parts,

$$d_t\langle u;\psi_j\rangle_{L^2} - \langle p,\nabla\cdot\psi_j\rangle_{L^2} + \int_{\partial\Omega} p\psi_j\cdot\hat{n}\,ds = 0.$$

Again the first term is exactly the $\psi_j$ of $u$, and the last term zero. The second term is rewritten using the eigenvalue problem (3.5) and manipulated so that it has the correct basis to decompose $p$,

$$\begin{aligned}\langle p,\nabla\cdot\psi_j\rangle &= \lambda_j\langle p,\bar{\sigma}\phi_j\rangle_{L^2} = \lambda_j\int_\Omega p\,\bar{\sigma}\,\phi_j\,dx = \lambda_j\int_\Omega p\,\frac{\phi_j}{\bar{\sigma}}\,\bar{\sigma}^2\,dx \\ &= \lambda_j\left\langle p,\frac{\phi_j}{\bar{\sigma}}\right\rangle_{w_1} = \lambda_j p_j,\end{aligned}$$

turning the first equation into

$$d_t u_j - \lambda_j p_j = 0.$$

We also manipulate the second equation in a very similar way to the second equation in (3.6). The first term is just rewritten $d_t\langle -v, \bar{\sigma}\phi_j\rangle_{w_2} = -d_t v_j$, and for the second term, we reverse the order of the manipulations, first changing the weight of the inner product and then integrating by parts,

$$\begin{aligned}\langle \nabla\cdot u, \bar{\sigma}\phi_j\rangle_{w_2} &= \left\langle \nabla\cdot u, \frac{\phi_j}{\bar{\sigma}}\right\rangle_{L^2} = \int_\Omega (\nabla\cdot u)\,\frac{\phi_j}{\bar{\sigma}}\,dx \\ &= -\int_\Omega u\cdot\nabla\Big(\frac{\phi_j}{\bar{\sigma}}\Big)\,dx + \int_{\partial\Omega}\frac{\phi_j}{\bar{\sigma}}u\cdot\hat{n}\,ds \\ &= \lambda_j\int_\Omega u\cdot\psi_j\,dx = \lambda_j\,\langle u;\psi_j\rangle_{L^2} = \lambda_j u_j.\end{aligned}$$

Thus, the second equation yields

$$-d_t v_j + \lambda_j u_j = 0,$$

and finally we write the system (3.13) as

$$\begin{cases} d_t u_j - \lambda_j p_j = 0 \\ d_t v_j - \lambda_j u_j = 0. \end{cases}$$

However, we cannot solve this system in the same way we did (3.8), nor should we expect to be able to do so. The equation of state $v = v(p,s)$ is generally nonlinear, so the $k$-mode components of our variables will depend on more than just the previous values of those $k$-mode components. This means there could never exist a system that respects modes like (3.8) for the nonlinear components. Although this system will be of use in a different way that we will see shortly.

## 3.5 The bifurcation problem

We now show that a solution to our problem will exist, the details of which will assist in designing an iterative method for finding such solutions. In particular, we show that periodic solutions of the nonlinear problem will exist, at least for sufficiently small $\alpha$.

We define the operator $\iota_2$ to lift $p(x,t_0)$ to a $u-p$ vector, with $u=0$, and the operator $\pi_1$ to project a $u-p$ vector onto its $u$-component, so

$$\iota_2 : p(x,t_0) \mapsto \begin{pmatrix} 0 \\ p(x,t_0)\end{pmatrix} \qquad \text{and} \qquad \pi_1 : \begin{pmatrix} u(x,t_0) \\ p(x,t_0)\end{pmatrix} \mapsto u(x,t_0).$$

Additionally, we define

$$\mathcal{E}^T : \begin{pmatrix} u(x,0) \\ p(x,0)\end{pmatrix} \mapsto \begin{pmatrix} u(x,T) \\ p(x,T)\end{pmatrix}$$

where $\big(u_T\,,\,p_T\big)^\intercal$ is found by nonlinearly evolving $\big(u_0\,,\,p_0\big)^\intercal$ according to (3.1) from $t=0$ to $t=T$. With these operators we write

$$f = \pi_1\mathcal{E}^T\iota_2$$

so that our nonlinear problem, finding a $p_0(x) = p(x,0)$ such that $u(x,T) = 0$, can be represented by just $f\big(p_0(x)\big) = 0$. We can write the nonlinear problem as

$$f\Big(\bar{p} + \alpha\frac{\phi_k}{\bar{\sigma}} + \sum_{j\neq k} a_j \frac{\phi_j}{\bar{\sigma}}\Big) = 0, \qquad \text{with} \qquad f(\bar{p}) = 0,$$

Viewing $\alpha$ as a parameter of the problem, we further rewrite the nonlinear problem as that of needing to solve

$$F(x,\alpha) = 0, \qquad \text{with} \qquad F(0,0) = 0.$$

So $F$ is a map

$$F : X \times \mathbb{R} \to Y, \qquad \text{where} \qquad X = H^b(\Omega)\backslash\tfrac{\phi_k}{\bar{\sigma}},\, Y = H^b(\Omega;\mathbb{R}^n).$$

With the assumption of nonresonance, we can decompose $x \in X$ as

$$x = z + w, \qquad \text{where} \qquad \ker\big\{D_xF(0,0)\big\} \oplus \ker\big\{D_xF(0,0)\big\}^\perp$$

where

$$z = a_0\frac{\phi_0}{\bar{\sigma}} \in \ker\big\{D_xF(0,0)\big\} \qquad \text{and} \qquad w = \sum_{j\neq 0,k} a_j\frac{\phi_j}{\bar{\sigma}} \in \ker\big\{D_xF(0,0)\big\}^\perp$$

are a 0-mode and a combination of all of the $j$-modes not in the kernel of $D_xF(0,0)$, which is $j \neq 0, k$.

If we define $\Pi_{\text{ran}}$ to be the projection onto the range of $D_xF(0)$,

$$\Pi_{\text{ran}} : Y \to \text{ran}\big\{D_xF(0,0)\big\}, \tag{3.14}$$

then we can rewrite the single equation

$$F(z+w,\alpha) = 0, \qquad \text{with} \qquad F(0,0) = 0$$

into an equivalent pair of equations consisting of the *auxiliary equation*

$$\Pi_{\text{ran}}F(z+w,\alpha) = 0 \tag{3.15}$$

and the *bifurcation equation*

$$(I - \Pi_{\text{ran}})F(z+w,\alpha) = 0.$$

We first solve the auxiliary equation, which for us is infinite dimensional, to find a

$$w = w(\alpha, z) \qquad \text{such that} \qquad F\big(z + w(\alpha,z),\alpha\big) = 0.$$

We can view the map (3.14) instead as the map

$$\Pi_{\text{ran}}F : \ker\big\{D_xF(0,0)\big\}^\perp \to \text{ran}\big\{D_xF(0,0)\big\},$$

in which case

$$D_x\Pi_{\text{ran}}F(0,0) : \ker\big\{D_xF(0,0)\big\}^{\perp} \to \text{ran}\big\{D_xF(0,0)\big\}$$

is an invertible map. Although we note that $D_x\Pi_{\text{ran}}F(0,0)$ is not bounded invertible on $\ker\big\{D_xF(0,0)\big\}^{\perp}$ because of the presence of the small divisors, the $\delta_j$'s, that we found in subsection 3.3.

It turns out, however, that with some work we can define a new norm such that $D_x\Pi_{\text{ran}}F(0,0)$ is indeed bounded invertible as a map from $\ker\big\{D_xF(0,0)\big\}^{\perp}$ to $\text{ran}\big\{D_xF(0,0)\big\}$. We start by just using the uniform $H^b$-norm, meaning for $p \in H^b(\Omega), u \in H^b(\Omega, \mathbb{R}^n)$ we write

$$\big|\big|p(x,t_0)\big|\big|_{H^b} = \sum_j (1+\lambda_j^2)^s p_j(t_0)^2 \qquad \text{with} \qquad p_j(t_0) = \Big\langle p(x,t_0), \frac{\phi_j}{\bar{\sigma}} \Big\rangle_{w_1=\bar{\sigma}^2}$$

and

$$\big|\big|u(x,t_0)\big|\big|_{H^b} = \sum_j (1+\lambda_j^2)^s u_j(t_0)^2 \qquad \text{with} \qquad u_j(t_0) = \Big\langle u(x,t_0); \frac{\psi_j}{\bar{\sigma}} \Big\rangle_{L^2}$$

respectively. Again referring back to subsection 3.3 and the small divisors,

$$D_x\Pi_{\text{ran}}F(0,0)\Big[\frac{\phi_j}{\bar{\sigma}}\Big] = \sin\Big(\frac{\lambda_j}{\lambda_k}\pi\Big)\psi_j = \delta_j\psi_j.$$

This means that for $j \neq 0, k$, which is to say for $j$ such that $\psi_j \in \text{ran}\big\{D_xF(0,0)\big\}$, we have

$$D_x\Pi_{\text{ran}}F(0,0)^{-1}\big[\psi_j\big] = \frac{1}{\delta_j}\frac{\phi_j}{\bar{\sigma}}.$$

We thus conclude that $D_x\Pi_{\text{ran}}F(0,0)$ is unbounded invertible, as it seems a $j$ can be found to make $\frac{1}{\delta_j}$ arbitrarily large, that is to say larger than any choice of $\hat{M} \in \mathbb{N}$, so long as we have enough time and will to go through the infinite set of $j$-modes until a $\delta_j$ is found that such that $\delta_j < \frac{1}{\hat{M}}$. Such was the case for the specific problems the computations in this dissertation focused on, but proving this, which might have to be done for a specific domain since the eigenvalue problem requires the domain be specified, seems to be more in the realm of number theory as thus lies outside the scope of this dissertation. However, $D_x\Pi_{\text{ran}}F(0,0)$ can be made bounded invertible if we define a new norm that scales along $j$-modes by $\delta_j^2$. Doing so, we are then able to invoke the implicit function theorem, which now guarantees a $w = w(\alpha, z)$ that solves (3.15).

Having found a $w(z,\alpha)$ such that $\Pi_{\text{ran}}F(z+w(\alpha,z),\alpha) = 0$, we move to solving the birfurcation equation for

$$z = z(\alpha) \qquad \text{such that} \qquad F\big(z(\alpha)+w(z(\alpha),\alpha),\alpha\big) = 0,$$

where now the implicit function theorem cannot be used since $\frac{\partial F}{\partial z}\big|_{(0,0)}$ is not invertible. We define a new function $G(z+w,\alpha)$ by

$$G(z+w(z,\alpha),\alpha) \coloneqq \begin{cases} \frac{1}{\alpha}F(z+w(z,\alpha),\alpha), & \alpha \neq 0 \\ \frac{\partial F}{\partial \alpha}(z+w(z,\alpha),0), & \alpha = 0. \end{cases}$$

The implicit function theorem can be applied to find a $z = z(\alpha)$ such that $G\big(z(\alpha) + w(z,\alpha),\alpha\big)$ so long as $\frac{\partial G}{\partial z}\big|_{(0,0)}$ is invertible. That is, we can solve the bifurcation provided

$$\frac{\partial G}{\partial z}\Big|_{(0,0)} \equiv \frac{\partial^2 F}{\partial z \partial \alpha}(0,0,0) \neq 0,$$

which is something we would not only expect to be the case because of the genuine nonlinearity of the problem but also show to be the case in the next section where we explicity find the above.

## 3.6 Solving the nonlinear problem

We now begin the process of designing a solution, continuing on with the assumption that we have solved the eigenvalue problem for each $\frac{\phi_j}{\bar\sigma}, \psi_j \lambda_j$ for some specified $\bar p$, $s$, and $\Omega$. We do already have the exact solution to the nonlinear problem in the stationary solution (3.3), although this is a disappointingly uninteresting a solution. We would obviously like to find a solution that is, at the very least, nonconstant.

To start towards this, we perturb the stationary solution (3.3) by our chosen $k$-mode, where we are making the choice of time period given by (3.11) so that the $k$-mode is a $2T$-periodic solution of the linearized problem. This initial solution, which we write as

$$p^{(0)}(x,0) = \bar p + (\alpha \bar p)\tfrac{1}{\bar\sigma}\phi_k, \tag{3.16}$$

where the superscript indicates this as the 0th iterate in a sequence of solutions, is unsuprisingly just an approximation of a periodic solution. This can be confirmed by directly checking that the residual,

$$\begin{aligned} f\big(p_0^{(0)}\big) &= f\Big(\bar p + (\alpha\bar p)\frac{\phi_k}{\bar\sigma}\Big) \\ &= f(\bar p) + Df(\bar p)\left[(\alpha\bar p)\frac{\phi_k}{\bar\sigma}\right] + \frac{1}{2}D^2 f(\bar p)\left[(\alpha\bar p)\frac{\phi_k}{\bar\sigma}, (\alpha\bar p)\frac{\phi_k}{\bar\sigma}\right] + \mathcal{O}(\alpha^3\bar p) \\ &= f(\bar p) + (\alpha\bar p)Df(\bar p)\left[\frac{\phi_k}{\bar\sigma}\right] + \frac{1}{2}(\alpha^2\bar p)D^2 f(\bar p)\left[\frac{\phi_k}{\bar\sigma}, \frac{\phi_k}{\bar\sigma}\right] + \mathcal{O}(\alpha^3\bar p) \\ &= \frac{1}{2}(\alpha^2\bar p)D^2 f(\bar p)\left[\frac{\phi_k}{\bar\sigma}, \frac{\phi_k}{\bar\sigma}\right] + \mathcal{O}(\alpha^3\bar p), \end{aligned} \tag{3.17}$$

is nonzero. We turn our attention momentarily to the second Fréchet derivative of our operator $f$, like the $D^2 f(\bar p)$ term in the above, as an understanding of this next Fréchet derivative will prove vital for actually solving the nonlinear problem.

The linearization of (3.1) around an arbitrary $p, u$ state of the form $\tilde p, 0$ can be written in the more general form

$$\begin{cases} \partial_t U + \partial_x P = 0 \\ \partial_t\left(-v_p(\tilde p, s)P\right) + \partial_x U = 0. \end{cases} \tag{3.18}$$

The Fréchet derivative of $f$ around $\tilde{p}$ is then the map

$$Df(\tilde{p}) : P(\cdot, 0) \mapsto U(\cdot, T),$$

and, because $f = \iota_2 \mathcal{E}^T \pi_1$, we can write $Df(\tilde{p}) = \iota_2 D\mathcal{E}^T \pi_1$. We thus describe the map $Df(\tilde{p})$ as lifting some provided initial data $P(x, 0)$ to the $P - U$ vector $\big(0\,,\, P(x,0)\big)^\intercal$, evolving this vector according to (3.18) to $t = T$, and projecting this vector onto the $U$-component $U(x, T)$.

We represent the linearization around $\tilde{p}, 0$ in a direction $P^{[i]}(x, 0) = \frac{\phi_i}{\bar{\sigma}}$ as the system

$$\begin{cases} \partial_t U^{[i]} + \partial_x P^{[i]} = 0 \\ \partial_t \left(-v_p(\tilde{p}, s) P^{[i]}\right) + \partial_x U^{[i]} = 0 \\ P^{[i]}(x, 0) = \frac{\phi_i}{\bar{\sigma}},\ U^{[i]}(x, 0) = 0, \end{cases} \tag{3.19}$$

consisting of both the governing equations and initial conditions. This is to make clear that the second linearization of (3.1) is found by linearizing the above in another, independent, direction $P^{[j]}(x, 0) = \frac{\phi_j}{\bar{\sigma}}$, where $P^{[j]}$ itself satisfies the analogous system

$$\begin{cases} \partial_t U^{[j]} + \partial_x P^{[j]} = 0 \\ \partial_t \left(-v_p(\tilde{p}, s) P^{[j]}\right) + \partial_x U^{[j]} = 0 \\ P^{[j]}(x, 0) = \frac{\phi_j}{\bar{\sigma}},\ U^{[j]}(x, 0) = 0. \end{cases}$$

So, linearizing (3.19) in the direction of $P^{[j]}$, we find the second linearization of (3.1) around $\tilde{p}, 0$ in direction $P^{[i]}(x, 0)$ and $P^{[j]}(x, 0)$ to be

$$\begin{cases} \partial_t U^{[i,j]} + \partial_x P^{[i,j]} = 0 \\ \partial_t \left(-v_p(\tilde{p}, s) P^{[i,j]}\right) + \partial_x U^{[i,j]} = \partial_t \left(v_{pp}(\tilde{p}, s) P^{[i]} P^{[j]}\right). \end{cases} \tag{3.20}$$

The second Fréchet derivative of $f$ around $\tilde{p}$ is the bilinear operator

$$D^2 f(\tilde{p}) : P^{[i]}(\cdot, 0) \times P^{[j]}(\cdot, 0) \mapsto U^{[i,j]}(\cdot, T).$$

Again, because we have defined $f = \iota_2 \mathcal{E}^T \pi_1$ and can thus write $D^2 f(\tilde{p}) = \iota_2 D^2 \mathcal{E}^T(\tilde{p}) \pi_1$, we find the action of this opreator can be described as

Finally, the last such operator that will be of use to us is the second Fréchet derivative of $f$ around $\tilde{p}, \tilde{u}$, which is the bilinear operator

$$D^2 f(\tilde{p}) : P^{[i]}(\cdot, 0) \times P^{[j]}(\cdot, 0) \mapsto U^{[i,j]}(\cdot, T).$$

In the above, $U^{[i,j]}(\cdot, T)$ is found by evolving the initial data $P^{[i,j]}(\cdot, 0) = 0$ and $U^{[i,j]}(\cdot, 0) = 0$ according to the second linearization around $\tilde{p}$,

$$\begin{cases} \partial_t U^{[i,j]} + \partial_x P^{[i,j]} = 0 \\ \partial_t \left(-v_p(\tilde{p}, s) P^{[i,j]}\right) + \partial_x U^{[i,j]} = \partial_t \left(v_{pp}(\tilde{p}, s) P^{[i]} P^{[j]}\right), \end{cases} \tag{3.21}$$

where $P^{[i]}$ and $P^{[j]}$ in the second equation are given by (3.18) and their respective initial data. However, what remains the same is that the whole nonlinear problem becomes a root finding problem with the periodicity condition $f(p_0) = 0$.

Our first approximation $p_0^{(0)}$ unsurprisingly has a nonzero residual,

$$\begin{aligned} f\big(p_0^{(0)}\big) &= f\Big(\bar{p} + (\alpha\bar{p})\frac{\phi_k}{\bar{\sigma}}\Big) \\ &= f(\bar{p}) + Df(\bar{p})\left[(\alpha\bar{p})\frac{\phi_k}{\bar{\sigma}}\right] + \frac{1}{2}D^2 f(\bar{p})\left[(\alpha\bar{p})\frac{\phi_k}{\bar{\sigma}}, (\alpha\bar{p})\frac{\phi_k}{\bar{\sigma}}\right] + \mathcal{O}(\alpha^3\bar{p}) \\ &= f(\bar{p}) + (\alpha\bar{p})Df(\bar{p})\left[\frac{\phi_k}{\bar{\sigma}}\right] + \frac{1}{2}(\alpha^2\bar{p})D^2 f(\bar{p})\left[\frac{\phi_k}{\bar{\sigma}}, \frac{\phi_k}{\bar{\sigma}}\right] + \mathcal{O}(\alpha^3\bar{p}) \\ &= \frac{1}{2}(\alpha^2\bar{p})D^2 f(\bar{p})\left[\frac{\phi_k}{\bar{\sigma}}, \frac{\phi_k}{\bar{\sigma}}\right] + \mathcal{O}(\alpha^3\bar{p}). \end{aligned} \tag{3.22}$$

Our goal is to reduce this residual with further perturbations of $p^{(0)}(x,0)$. To ensure we will reach a nontrivial solution and not disappointingly wind up back at a stationary solution, we fix the chosen $k$-mode term in (3.16) throughout this whole process and instead only allow make further perturbations with other $j$-modes, $j \neq k$. So, each step of our iteration will be of the form

$$p_0^{(i+1)} = p_0^{(i)} + \sum_{j\neq k} r_j^{(i)} \frac{\phi_j}{\bar{\sigma}}. \tag{3.23}$$

## 3.7 Quasi-Newton methods

We design quasi-Newton methods by, similarly to (2.19), expanding $f$ around an arbitrary $p_0^{(i+1)}$,

$$\begin{aligned} f\big(p_0^{(i+1)}\big) &= f\Big(p_0^{(i)} + \sum_{j\neq k} r_j^{(i)} \frac{\phi_j}{\bar{\sigma}}\Big) \\ &= f\big(p_0^{(i)}\big) + \sum_{j\neq k} r_j^{(i)} Df\big(p_0^{(i)}\big)\Big[\frac{\phi_j}{\bar{\sigma}}\Big] \\ &+ \frac{1}{2}\sum_{j\neq k}\sum_{l\neq k} r_j^{(i)} r_l^{(i)} D^2 f\big(p_0^{(i)}\big)\Big[\frac{\phi_j}{\bar{\sigma}}, \frac{\phi_l}{\bar{\sigma}}\Big] + \mathcal{O}(r^3) \\ &= f\big(p_0^{(i)}\big) + \sum_{j\neq k} r_j^{(i)} Df\big(\bar{p}\big)\Big[\frac{\phi_j}{\bar{\sigma}}\Big] + \sum_{j\neq k} r_j^{(i)} D^2 f(\bar{p})\Big[p_0^{(i)} - \bar{p}, \frac{\phi_j}{\bar{\sigma}}\Big] \\ &+ \frac{1}{2}\sum_{j\neq k}\sum_{l\neq k} r_j^{(i)} r_l^{(i)} D^2 f\big(p_0^{(i)}\big)\Big[\frac{\phi_j}{\bar{\sigma}}, \frac{\phi_l}{\bar{\sigma}}\Big] + \mathcal{O}(\alpha^2 r) + \mathcal{O}(r^3) \end{aligned}$$

Again many different quasi-Newton methods can be designed from this expansion, but our computations using evolution in time parimarily focus on the same two methods we used when evolving in space,

$$p_0^{(i+1)} = p_0^{(i)} - \Big(Df(\bar{p}) + D^2 f(\bar{p})[\bar{p}]\Big)^{-1}\Big[f\big(p^{(i)}\big)\Big] \tag{3.24}$$

and the faster but less accurate

$$p^{(i+1)} = p^{(i)} - Df(\bar{p})^{-1}\Big[f\big(p^{(i)}\big)\Big]. \tag{3.25}$$

Implementing either of these two methods brings up an issue, the ceaseless hunger for a computer's resources that infinite-dimensional computations possess. So, we must make an approximation and use finite subsets of the infinite-dimensional eigenfunction bases for $p$ and $u$ as their respective bases.

Taking $N$ as the size to limit the eigenfunction bases to, we have the $p$-basis $\{\frac{1}{\bar{\sigma}}\phi_j\}_{j=0,\dots N}^{j\neq k}$ and the $u$-basis $\{\psi_j\}_{j=1,\dots,N}^{j\neq 0}$. Again the $k$-mode is fixed, so $\frac{1}{\bar{\sigma}}\phi_k$ is omitted from the $p$-basis, and since $\psi_0 = 0$ it would be pointless to include in the $u$-basis. These bases grant the same advantage we wanted in the previous section, the advantage finite-dimensional computations always have over their infinite-dimensional counterparts, namely the possibility of finishing running. We write $p$ and $u$ as vectors over these finite eigenfunction bases, expressing

$$p^{(i)}(x,t) = \sum_{j\neq k} p_j^{(i)}(t)\frac{\phi_j}{\bar{\sigma}} \quad \text{and} \quad u^{(i)}(x,t) = \sum_{j\neq 0} u_j^{(i)}(t)\psi_j,$$

instead as the vectors

$$p_0^{(i)} = \begin{pmatrix} p_0^{(i)}(0) \\ \cdots \\ p_{k-1}^{(i)}(0) \\ p_{k+1}^{(i)}(0) \\ \cdots \\ p_N^{(i)}(0) \end{pmatrix} \quad \text{and} \quad u_T^{(i)} = f\big(p^{(i)}\big) = \begin{pmatrix} u_1^{(i)}(T) \\ \cdots \\ u_{k-1}^{(i)}(T) \\ u_k^{(i)}(T) \\ \cdots \\ u_N^{(i)}(T) \end{pmatrix}.$$

As in the previous method, we express the operator $Df(\check{p})$ as a matrix by writing

$$Df(\check{p}) = \Big(Df(\check{p})\big[\tfrac{1}{\bar{\sigma}}\phi_0\big] \quad \cdots \quad Df(\check{p})\big[\tfrac{1}{\bar{\sigma}}\phi_{k-1}\big] \quad Df(\check{p})\big[\tfrac{1}{\bar{\sigma}}\phi_{k+1}\big] \quad \cdots \quad Df(\check{p})\big[\tfrac{1}{\bar{\sigma}}\phi_N\big]\Big),$$

where the $n$th column is the $u$-vector image found by passing $Df(\check{p})$ the $n$th $p$-basis element $\frac{1}{\bar{\sigma}}\phi_n$. With these, we can make better sense of (3.25) and (3.24). As we previously found, we can project the first equation of (3.1) onto $\psi_j$ and the second equation onto $\bar{\sigma}\phi_j$ and with some manipulation find

$$\begin{cases} d_t u_j - \lambda_j p_j = 0 \\ d_t v_j - \lambda_j u_j = 0, \end{cases} \tag{3.26}$$

which can be used to design many finite-difference schemes, but we opt to not go further into the design of the numerical scheme using time evolution here, instead covering this in the next section.

## 4 Time periodic solutions on the disc

For our first computational examples utilizing the time evolution method, we will begin with the two dimensional domain $\Omega = B_0^\nu$, which is to say the disc of radius $\nu$, which is a choice we can make without loss of generality, centered on the origin. We begin with this domain rather than one that is one dimensional

as this is the simplest case that also allows for us to solve the isentropic problem. As with the one dimensional computations using the spatial evolution method, all of the computations featured in this section utilize for their equation of state that from assuming an ideal gas and a reversible process,

$$v(p,s) = \tilde{v}\left(\frac{p}{\tilde{p}}\right)^{-\frac{1}{\gamma}} e^{-\frac{s-\tilde{s}}{c_p}}, \tag{4.1}$$

which in the isentropic case becomes

$$v(p) = \bar{v}\left(\frac{p}{\bar{p}}\right)^{-\frac{1}{\gamma}}. \tag{4.2}$$

The parameters within our code for computing solutions are in general chosen so as to best approximate the behavior of air in a real world setting. We demonstrate what this means for us in the more general case of having to use (4.1). We assume the background state of the disc to be one of constant atmospheric background pressure,

$$\bar{p} = 101325 \text{ bar},$$

and $L$ annular regions of constant temperature. That is, there is assumed to be a radially piecewise constant temperature with $L-1$ jumps between $L$ distinct temperatures, characterized by the points $r_0, r_1, \ldots, r_{L-1}$ where $r_0 = 0$ and $r_{L-1} = \nu$,

$$T(r) = \begin{cases} T_0 & 0 < r < r_1 \\ T_1 & r_1 < r < r_2 \\ \cdots & \\ T_{L-1} & r_{L-2} < r < 1. \end{cases} \tag{4.3}$$

Rather than just requiring the temperatures to be within the range a human can survive, we choose to use values between 50°F and 80°F. Since the background pressure is the same across the disc, we only know how one single variable changes annulus to annulus, and turning (4.3) into a piecewise constant entropy profile requires knowing how one additional variable changes. To get this additional information we use data tables to find the measured specific volume of air at atmospheric pressure and varying temperatures. Since we have assumed an ideal gas knowing both $T_0, \ldots, T_{L-1}$ and $v_0, \ldots, v_{L-1}$ suffices to determine the entropy in each region. Taking the innermost region to be the reference state, we also look up values in data tables for the specific heat capacities $c_p$ and $c_v$ for air at atmospheric pressure and temperature $T_0$. These values also determine $R$ and $\gamma$ by

$$R = c_p - c_v \quad \text{and} \quad \gamma = \frac{c_p}{c_v}.$$

Now, choosing to set the entropy in the reference region to be zero, $s_0 = 0$, we calculate the entropy in each of the other $L-1$ annular regions

$$s_i - s_0 = c_v \log\left(\frac{T_i}{T_0}\right) + R \log\left(\frac{v_i}{v_0}\right)$$

and thus find an entropy profile

$$s(r) = \begin{cases} 0 & 0 < r < r_1 \\ s_1 & r_1 < r < r_2 \\ \cdots & \\ s_{L-1} & r_{L-2} < r < 1. \end{cases}$$

The parameters of the problem now taken care of, we move to the biggest task that depends on a specified domain to complete, namely finding the eigenfunction bases.

## 4.1 Solving the Helmholtz equation on the isentropic disc

We first solve the eigenvalue problem for the simpler isentropic case. We can view this as the particular case of (4.3) where the entire disc is specified to be at the same temperature, and with no differences in temperature across the domain, we have

$$s = 0.$$

Since $s$ is constant, $\bar{\sigma} = \sigma(\bar{p}, s)$ is constant on the entire disc as well, and because $\bar{\sigma}$ is constant, the differential equation

$$-\Delta\left(\frac{\phi_j}{\bar{\sigma}}\right) = \bar{\sigma}^2\lambda_j^2\left(\frac{\phi}{\bar{\sigma}}\right)$$

is valid everywhere. That is to say, there is no worry about matching conditions within the disc as there would be with an entropy profile featuring jumps. We note that although $u$ has its own eigenfunction basis $\{\psi_j\}$, our computations only make direct use of the $p$-basis $\{\frac{\phi_j}{\bar{\sigma}}\}$ and $v$-basis $\{\bar{\sigma}\phi_j\}$. Writing the Laplacian in polar coordinates and separating variables by $\frac{\phi}{\bar{\sigma}} = R(r)\Theta(\theta)$, the above expression becomes

$$-\frac{r^2R''(r)+rR'(r)+r^2\bar{\sigma}^2\lambda^2R(r)}{R(r)} = \frac{\Theta''(\theta)}{\Theta(\theta)} = -n^2.$$

Thus,

$$\Theta_n(\theta) = a_n\cos(n\theta) + b_n\sin(n\theta), \tag{4.4}$$

and for the radial component we have

$$r^2R''(r) + rR'(r) + \left(r^2\bar{\sigma}^2\lambda_j^2 - n^2\right)R(r) = 0.$$

This becomes a Bessel equation of order $n$ with the substitution $\rho = \bar{\sigma}\lambda r$,

$$\rho^2R''(\rho) + \rho R'(\rho)\left(\rho^2 - n^2\right)R(\rho) = 0,$$

so

$$R_{n,m}(r) = J_n(\bar{\sigma}\lambda_{n,m}r). \tag{4.5}$$

In the above, $J_n$ is the $n$th order Bessel function of the first kind. The $n$th order Bessel function of the second kind would also satisfy the differential equation,

but all Bessel functions of the second kind have a singularity at the origin and so conflict with our ultimate goal of finding smooth, periodic solutions. With our separation of variables, the eigenvalue problem's boundary condition $\nabla\phi\cdot\hat{n}=0$ becomes $J_n'(\bar{\sigma}\lambda\nu)=0$, so we find the eigenvalue in (4.5) is given by

$$\lambda_{n,m}=\frac{j'_{n,m}}{\bar{\sigma}\nu}, \tag{4.6}$$

where we denote by $j'_{n,m}$ the $m$th zero of $J_n'(r)$.

These zeros are not only used for any computation on the isentropic disc but also for any computation on the nonisentropic disc and either case for the annulus. Because these values are needed for all of the 2D domains we work on in this dissertation, we opt to create a data table of values of $j'_{n,m}$ to simply import at runtime. There are publications available that include tables of values for the zeros of Bessel functions and their derivatives, but we would like to have values for $j'n,m$ with $n$ and $m$ several times greater the limits these tables usually have. Each $J_n(r)$ is oscillatory, so to use some root-finding algorithm we need to have some idea of what $j'_{n,m}$ is for any given $n$ and $m$. For this we use the asymptotic expansion McMahon wrote in 1894 [6],

$$j'_{n,m}\sim b-\frac{\mu+3}{8b}-\frac{4(7\mu^2+82\mu-9)}{3(8b)^3}-\frac{32(83\mu^3+2075\mu^2-3039\mu+3537)}{15(8b)^5}$$

with

$$\mu=4n^2,\qquad b=m+\frac{1}{2}n-\frac{3}{4}.$$

With this, we use a basic bisection algorithm to compute $j'_{n,m}$ for $n=0,\dots,200$ and $m=1,\dots,200$ within a tolerance of $10^{-20}$.

We can now construct both the $p$-basis and $v$-basis eigenfunction bases with (4.4), (4.5), and (4.6). We find the $p$-basis elements are given by

$$\frac{\phi_{n,m}}{\bar{\sigma}}=R(r)\Theta(\theta)=J_n(\bar{\sigma}\lambda_{n,m}r)\big(a_{n,m}\cos(n\theta)+b_{n,m}\sin(\sin(n\theta))\big), \tag{4.7}$$

with the $v$-basis elements being found by simply multiplying the above by $\bar{\sigma}^2$. We note at this time that we make the choice, for all of our computations, that

$$a_{n,m}=1,\qquad b_{n,m}=0 \tag{4.8}$$

for all $n,m$. The particular choice made for the pairs of coefficients $a_{n,m}$ and $b_{n,m}$ are not all too important, and making the choice above is just the simplest one can make that leads to a basis that is closed under multiplication. That is, (4.8) gives a basis such that for any $\frac{\phi_{n_1,m_1}}{\bar{\sigma}}$ and $\frac{\phi_{n_2,m_2}}{\bar{\sigma}}$ we have

$$\frac{\phi_{n_1,m_1}}{\bar{\sigma}}\frac{\phi_{n_2,m_2}}{\bar{\sigma}}\in\Big\{\sum_{n,m}c_{n,m}\frac{\phi_{n,m}}{\bar{\sigma}}\mid c_{n,m}\in\mathbb{R}\Big\}$$

One may also take notice of the lack of any mention of the $u$-basis eigenfunctions $\psi_{n,m}$. This is simply because we will not directly make use of the $u$-basis, using instead only its decomposition coefficients $u_{n,m}$, but if one wanted to reconstruct the actual velocity field from these coefficients, the $u$ basis can be found by (4.7) along with (3.5).

## 4.2 Eigenfunction bases on the isentropic disc

We now deal with the issues of how many and which elements of the infinite dimensional basis $\{\frac{\phi_{n,m}}{\bar{\sigma}}\}$ to use and what kind of representation we will store for each $\frac{\phi_{n,m}}{\bar{\sigma}}$. We start with the question of what kind of representation of the eigenfunctions to use. We naturally will want to represent the eigenfunctions as a discrete matrix of values on a discretization of the polar axes $[0, 2\pi] \times [0, R]$, but there are two different kinds of discretizations of the polar axes that we used for this. There is of course the option of using a uniform discretization of $[0, 2\pi] \times [0, R]$,

$$\{\chi_{q,w}\} = \big\{ \, (q\Delta\theta, w\Delta r) \mid 0 \leq q \leq Q, 0 \leq w \leq W \, \big\}, \tag{4.9}$$

although this comes with the downside of points being spread more sparsely as the radius increases. We make use of such a discretization for some computations despite this because it does allow us to make use of a simple yet fairly fast and accurate numerical integration method like Simpson's 1/3 rule, which is the method used for all of our computations on a uniform discretization. To compensate in some way for the inaccuracies introduced with points being packed more densely around the origin, only very fine uniform discretizations of the disc were used. In particular, these discretizations ranged from $Q = 200, W = 200$ to $Q = 400, W = 500$.

We can improve the uniform discretization's uneven clustering by instead using a nonuniform discretization. There are countless possibilities for a nonuniform discretization, but we limit ourselves to two cases of a nonuniform radial grid paired with a uniform angular grid. One such case we use it that with an exponential radial grid, which means a discretization of the form

$$\{\chi_{q,w}\} = \Big\{ \Big( q\Delta\theta \, , \, \nu \, \frac{1 - e^{s \frac{w\Delta r}{\nu}}}{1 - e^{s}} \Big) \mid 0 \leq q \leq Q, 0 \leq w \leq W \Big\}. \tag{4.10}$$

In the above, $s \in \mathbb{R}$ is a scaling factor that affects which end of the interval $[0, \nu]$ points are clustered towards and the degree to which they are clustered. The other nonunfirom radial grid used in some of our computations is a polynomial grid, with the discretization then being

$$\{\chi_{q,w}\} = \Big\{ \Big( q\Delta\theta \, , \, \nu \Big( \frac{w}{W} \Big)^{\varepsilon} \Big) \mid 0 \leq q \leq Q, 0 \leq w \leq W \, \big\}. \tag{4.11}$$

When using either (4.10) or (4.11), we just switch our numerical integration method to the trapezoid rule, rather than a much more complicated version of Simpson's rule that allows for nonuniform spacing.

We now turn to the issue of what finite subset of the infinite dimensional eigenfunction basis will serve as our basis. There is a bit more subtlety here than may be evident at first glance, as it turns out to be naive, or at least computationally wasteful, to use a finite subset of the form

$$\Big\{\frac{\phi_{0,0}}{\bar{\sigma}}\Big\} \cup \Big\{\frac{\phi_{n,m}}{\bar{\sigma}} \mid 0 \leq n \leq N, 1 \leq m \leq M\Big\}.$$

To see why we start by giving our chosen $k$-mode two specific indices $\zeta$ and $\eta$,

$$\frac{\phi_k}{\bar{\sigma}} = \frac{\phi_{\zeta,\eta}}{\bar{\sigma}} = \frac{1}{\bar{\sigma}} J_\zeta(\bar{\sigma}\lambda_{\zeta,\eta} r)\cos(\zeta\theta),$$

and looking back to (3.22), the expansion of $f$ at the starting $p_0^{(0)}(x)$ about the background state $\bar{p}$. This expansion can be seen as being composed of the effects from the $k$-mode interacting with itself, the $k$-mode interacting with the result of that interaction, and so on. Earlier we used Duhamel's principle to turn the problem of solving for

$$\frac{1}{2} D^2 f(\bar{p})\Big[\frac{\phi_{\zeta,\eta}}{\bar{\sigma}}, \frac{\phi_{\zeta,\eta}}{\bar{\sigma}}\Big] \tag{4.12}$$

into a family of problems with initial data

$$\begin{aligned} P^\varepsilon(x,\varepsilon) &= -v_{pp}(\bar{p},s)\partial_t\Big[\Big(\frac{\phi_{\zeta,\eta}}{\bar{\sigma}}\cos(\lambda_{\zeta,\eta}t)\Big)\Big(\frac{\phi_{\zeta,\eta}}{\bar{\sigma}}\cos(\lambda_{\zeta,\eta}t)\Big)\Big|_{t=\varepsilon} \\ &= -v_{pp}(\bar{p},s)\lambda_{\zeta,\eta}\sin(2\lambda_{\zeta,\eta}\varepsilon)\Big(\frac{\phi_{\zeta,\eta}}{\bar{\sigma}}\Big)^2. \end{aligned}$$

This means that (4.12) will be composed of $(n,m)$-modes that satisfy

$$\Big\langle\Big(\frac{\phi_{\zeta,\eta}}{\bar{\sigma}}\Big)^2, \frac{\phi_{n,m}}{\bar{\sigma}}\Big\rangle_{w_1} \neq 0.$$

For the next term,

$$\frac{1}{6} D^3 f(\bar{p})\Big[\frac{\phi_{\zeta,\eta}}{\bar{\sigma}}, \frac{\phi_{\zeta,\eta}}{\bar{\sigma}}, \frac{\phi_{\zeta,\eta}}{\bar{\sigma}}\Big], \tag{4.13}$$

Duhamel's principle has us solve a family of problems with initial data (B.7), and we similarly find (4.13) is composed of $(n,m)$-modes that satisfy

$$\Big\langle\Big(\frac{\phi_{\zeta,\eta}}{\bar{\sigma}}\Big)^2, \frac{\phi_{n,m}}{\bar{\sigma}}\Big\rangle_{w_1} \neq 0 \quad \text{or} \quad \Big\langle\Big(\frac{\phi_{\zeta,\eta}}{\bar{\sigma}}\Big)^3, \frac{\phi_{n,m}}{\bar{\sigma}}\Big\rangle_{w_1} \neq 0.$$

Continuing on in this way, we conclude that our basis should only be comprised of $(n,m)$-modes for which there exists some $q \in \mathbb{N}$ such that

$$\Big\langle\Big(\frac{\phi_{\zeta,\eta}}{\bar{\sigma}}\Big)^q, \frac{\phi_{n,m}}{\bar{\sigma}}\Big\rangle_{w_1} \neq 0, \tag{4.14}$$

as they would not be used in correcting our residual if this does not hold. Using the expression found earlier for the eigenfunctions on the isentropic disc (4.7),

we see

$$\begin{aligned}\Big(\frac{\phi_{\zeta,\eta}}{\bar{\sigma}}\Big)^q &= \Big[J_\zeta(\bar{\sigma}\lambda_{\zeta,\eta}r)\cos(\zeta\theta)\Big]^q = \Big[J_\zeta(\bar{\sigma}\lambda_{\zeta,\eta}r)\Big]^q\Big[\frac{1}{2}\Big(1+\cos(2\zeta\theta)\Big)\cos^{q-2}(\zeta\theta)\Big] \\ &= \Big[J_\zeta(\bar{\sigma}\lambda_{\zeta,\eta}r)\Big]^q\Big[\frac{1}{4}\Big(3\cos(\zeta\theta)+\cos(3\zeta\theta)\Big)\cos^{q-3}(\zeta\theta)\Big] \\ &= \Big[J_\zeta(\bar{\sigma}\lambda_{\zeta,\eta}r)\Big]^q\Big[\frac{1}{8}\Big(3+4\cos(2\zeta\theta)+\cos(4\zeta\theta)\Big)\cos^{q-4}(\zeta\theta)\Big] \\ &= \Big[J_\zeta(\bar{\sigma}\lambda_{\zeta,\eta}r)\Big]^q\Big[\frac{1}{16}\Big(10\cos(\zeta\theta)+5\cos(3\zeta\theta)+\cos(5\zeta\theta)\Big)\cos^{q-5}(\zeta\theta)\Big],\end{aligned}$$

and we can continue on like this, multiplying the inside expression by one more cosine at a time. In this way, we find

$$\begin{aligned}\Big(\frac{\phi_{\zeta,\eta}}{\bar{\sigma}}\Big)^q &= \cos(0)\sum_{m=0} c_{0,m}J_0(\bar{\sigma}\lambda_{0,m}r) + \cos(2\zeta\theta)\sum_{m=1} c_{2,m}J_2(\bar{\sigma}\lambda_{2\zeta,m}r) + \dots \\ &\quad + \cos(q\zeta\theta)\sum_{m=1} c_{q,m}J_q(\bar{\sigma}\lambda_{q\zeta,m}r)\end{aligned}$$

for $q$ even and

$$\begin{aligned}\Big(\frac{\phi_{\zeta,\eta}}{\bar{\sigma}}\Big)^q &= \cos(\zeta\theta)\sum_{m=0} c_{1,m}J_1(\bar{\sigma}\lambda_{1,m}r) + \cos(3\zeta\theta)\sum_{m=1} c_{3,m}J_3(\bar{\sigma}\lambda_{3\zeta,m}r) + \dots \\ &\quad + \cos(q\zeta\theta)\sum_{m=1} c_{q,m}J_q(\bar{\sigma}\lambda_{q\zeta,m}r)\end{aligned}$$

Based on many numerical tests, it seems that one can assume that all of the $c_{n\zeta,m}$ coefficients are nonzero. There are publications about integrals of the products of three Bessel functions, which is the kind of integral that one would need to solve to find these coefficients, but expressions for such integrals involve hypergeometric functions and are outside the scope of this dissertation.

A finite basis limited to $(n,m)$-modes that satisfy (4.14) is thus of the form

$$\Big\{\frac{\phi_{0,0}}{\bar{\sigma}}\Big\} \cup \Big\{\frac{\phi_{n\zeta,m}}{\bar{\sigma}} \mid 0 \le n \le N-1, 1 \le m \le M\Big\}. \tag{4.15}$$

The choices made for $N$ and $M$ are among the most consequential choices for how long a computation will take to run, along with the choices made for how many points the discretization will be. The computations featured in this dissertation use bases of the the form (4.15) with $N = 28, M = 20$ on the higher end.

## 4.3 A scheme for evolution in time

There are now just two more parts of the entire computational scheme as a whole that we must develop before moving to look at any results. Of these remaining parts, we start with the need for a time evolution scheme. We need a

time evolution scheme to be able to calculate the residual $f\big(p_0(x)\big)$, where the decomposition of the initial data

$$p_0(x) = p_{0,0}(0)\frac{\phi_{0,0}}{\bar{\sigma}} + \sum_{n=0}^{N-1}\sum_{m=1}^{M} p_{n\zeta,m}(0)\frac{\phi_{n\zeta,m}(x)}{\bar{\sigma}}$$

is known, in order to be able to iteratively apply some quasi-Newton step like (3.24) or (3.25).

We do this similarly to the spatial evolution case by evolving the decomposition coefficients $p_{n\zeta,m}$ and $u_{n\zeta,m}$, which we also know at $t = 0$ since $u(x, 0) = 0$, via a finite-difference scheme, meaning we again use a mixed spectral finite-difference method. Projecting each equation in (3.1) onto appropriate eigenfunctions, we found (3.26). One can design many different finite-difference schemes from this system, but we again opt to use a leapfrog type scheme,

$$\begin{cases} u^i_{n\zeta,m} = u^{i-2}_{n\zeta,m} + (2\Delta t\lambda_{n\zeta,m})p^{i-1}_j \\ v^i_{n\zeta,m} = v^{i-2}_{n\zeta,m} + (2\Delta t\lambda_{n\zeta,m})u^{i-1}_j, \end{cases} \tag{4.16}$$

where the superscript denotes the timestep, $\xi_i$, in the uniform $D$-point discretization of $[0, T]$,

$$\{\xi_i\} = \Big\{ i\Delta t \,|\, 0 \le i \le D, \Delta t = \frac{T}{D}\Big\},$$

and the subscript again denotes these as coefficients of the $j$-mode eigenfunctions,

$$\Big\langle p(x, \xi_i), \frac{\phi_{n\zeta,m}}{\bar{\sigma}} \Big\rangle_{w_1} = p^i_{n\zeta,m}.$$

Two-step methods like (4.16) require two pieces of initial data. Namely, we need information at $\xi_1 = \Delta t$ in order to run this scheme. We find this second piece of initial data by evolving $p^0_{n\zeta,m}$ and $u^0_{n\zeta,m} = 0$, the initial data that we do have, according to a simple forward Euler step

$$\begin{cases} u^\tau_{n\zeta,m} = u^{\tau-1}_{n\zeta,m} + (\frac{\Delta t}{E}\lambda_{n\zeta,m})p^{\tau-1}_{n\zeta,m} \\ v^\tau_{n\zeta,m} = v^{\tau-1}_{n\zeta,m} + (\frac{\Delta t}{E}\lambda_{n\zeta,m})u^{\tau-1}_{n\zeta,m} \end{cases} \tag{4.17}$$

where now the superscript $\tau$ denote the timestep in the $E$-point discretization of the subinterval $[0, \Delta t]$,

$$\{\Xi_\tau\} = \Big\{ \tau\frac{\Delta t}{E} \,|\, 0 \le \tau \le E\Big\}.$$

The Euler step has accuracy $\mathcal{O}\big(\frac{\Delta t}{E}\big)$, in contrast to the $\mathcal{O}(\Delta t)^2$ accuracy of (4.16), so the number of points $E$ in the discretization of $[0, \Delta t]$ can be chosen larger to mitigate any loss in accuracy that may come from using (4.17) to extend the initial data.

We now note the fact that (4.16) and (4.17) use the decomposition coefficients of $v$. We start only knowing the $p^0_{n\zeta,m}$'s, and $u^0_{n\zeta,m}$'s since $u(x,0)=0$, so we need to find the $v^0_{n\zeta,m}$'s from the $p^0_{n\zeta,m}$'s in order to be able to start this time evolution scheme. There is of course more than one method one can employ for this task. The first such method employed in the course of working on this dissertation is the direct approach of applying the equation of state (4.2) pointwise to the array of values

$$p(\chi_{q,w},0)=p^0_{0,0}\frac{\phi_{0,0}(\chi_{q,w})}{\bar{\sigma}}+\sum_{n=0}^{N-1}\sum_{m=1}^{M}p^0_{n\zeta,m}\frac{\phi_{n\zeta,m}(\chi_{q,w})}{\bar{\sigma}}$$

to find

$$v(\chi_{q,w},0)=\bar{v}\left(\frac{p(\chi_{q,w},0)}{\bar{p}}\right)^{-\frac{1}{\gamma}}$$

and then using this array to approximate

$$\langle v(x,0)\,,\,\bar{\sigma}\phi_{n\zeta,m}\rangle_{w_2}=\int_{B_0^\nu}v(x,0)(\bar{\sigma}\phi_{n\zeta,m})w_2(x)\,dx$$

by numerical integration. Our computations that handled decomposition in this way utilized Simpson's 1/3 rule for a uniform discretization (4.9), and the trapezoidal rule for nonuniform discretizations like (4.10) and (4.11).

Although we cannot make direct use of the FFT for decomposition as done in the spatial evolution method, since our eigenfunction basis is no longer a simple Fourier basis, all hope is not completely lost. The difficulty of decomposing $v(x,0)$ or $p(x,\xi_i)$ primarily comes from the radial integral, which is of the form of a Hankel transform,

$$F_\mu(k)=\int f(r)J_\mu(kr)\,r\,dr.$$

One can employ the discrete Hankel transform for approximating such integrals, independently developed first by Johnson [5] and then again by Yu et al. [12] for $\mu=0$ and Guizar-Sicairos [4] for $\mu\in\mathbb{N}$. However, there is actually a better suited transform for us to use in this case yet. The eigenfunctions we are using, $\cos(n\theta)J_n(\lambda_{n,m}\bar{\sigma}r)$ or $\sin(n\theta)J_n(\lambda_{n,m}\bar{\sigma}r)$, are particular component of the basis functions for the two-dimensional polar Fourier transform, $e^{in\theta}J_n(k_{n,m}r)$. So, we can find our decompositions with the relatively recent work of Baddour on the discrete 2D polar FFT [2], which relates the polar decomposition coefficients to those found by zero padding the data on the disc to the square and then performing the regular 2D Cartesian FFT. More details on the specific relationship between these coefficients can be found in appendix C.

We also note here that once we use any decomposition method to find the $v^0_j$'s from the $p^0_j$'s, we actually face the reverse problem at every other timestep $\xi_i$. Namely, after applying (4.16) to find the $u^i_j$'s and $v^i_j$'s, or after applying (4.17) to find the $u^1_j$'s and $v^1_j$'s specifically, we need to find the $p^i_j$'s to continue

the iteration. This is done in the same manner that we found the $v_j^0$'s from the $p_j^0$'s at the start. That is, we recompose

$$v(\chi_{q,w}, \xi_i) = v_{0,0}^i\big(\bar{\sigma}\phi_{0,0}(\chi_{q,w})\big) + \sum_{n=0}^{N-1}\sum_{m=1}^{M} v_{n\zeta,m}^i\big(\bar{\sigma}\phi_{n\zeta,m}(\chi_{q,w})\big)$$

to find

$$p(\chi_{q,w}, \xi_i) = \bar{p}\left(\frac{v(\chi_{q,w}, \xi_i)}{\bar{v}}\right)^{-\gamma}$$

and then approximate the decomposition coefficients

$$\left\langle p(x, \xi_i)\,,\, \frac{\phi_{n\zeta,m}}{\bar{\sigma}}\right\rangle_{w_1}$$

by either numerical integration or by using the 2D Cartestian FFT coefficients, whichever method was used for finding the $v_j^0$'s.

## 5 Time periodic solutions on the line

Here we continue looking at solutions of the Euler equations in one spatial dimension, and so satisfying

$$\begin{cases} \partial_t u + \partial_x p = 0 \\ \partial_t v - \partial_x u = 0, \end{cases}$$

dgb vc x

# Appendices

## A Fréchet derivatives and evolution operators

In this appendix we cover the details of Fréchet derivatives, in particular the Fréchet derivatives of evolution operators for differential equations, which is key to the development of all of our iteration methods throughout this dissertation.

We let $X, Y$ be Hilbert spaces, $\Omega$ an open subset of $X$, and $f$ a mapping from $\Omega$ to $Y$. We say that $f$ is *Gâteaux-differentiable* at $u_o \in \Omega$ in the direction of $v \in X$ if the limit

$$d_v f(u_0) \coloneqq \lim_{\delta \to 0} \frac{f(u_0 + \delta v) - f(u_0)}{h}$$

exists, in which case $d_v f(u_0)$ is called the *Gâteaux derivative* of $f$ at $u_0$ in the direction of $v$. In the case of a scalar function $f : \mathbb{R}^n \mapsto \mathbb{R}$ this is equivalent to the usual notion of a directional derivative, and it is in this way that the Gâteaux derivative can be seen as a generalization of the directional derivative,

one that can be used in arbitrary, possibly infinite-dimensional as in our case, topological spaces. Of more use for us is the following more restrictive idea of differentiation that generalizes the Jacobian to arbitrary topological spaces. A map $f : X \mapsto Y$ is said to be *Fréchet-differentiable* at $u_0 \in \Omega$ if there exists a bounded linear map $Df(u_0)$ such that

$$\lim_{|v|_X \to 0} \frac{\big|f(u_0+v) - f(u_0) - Dg(u_0)[v]\big|_Y}{|v|_X} = 0$$

or, equivalently,

$$\big|f(u_0+v) - f(u_0) - Df(u_0)[v]\big|_Y = o\big(|v|_X\big).$$

If this is the case, then we call $Df(u_0) \in \mathcal{B}(\Omega, Y)$, the best linear approximation of $f$ centered around the point $u_0$, the *Fréchet derivative* of $f$ around $u_0$.

We define

$$\begin{aligned}
\Delta f(u,v)[v] &\coloneqq f(u+v) - f(u),\\
\Delta^2 f(u,v)\big[v^{[2]}\big] &\coloneqq \Delta f(u+v)[v] - Df(u)[v],\\
&\cdots,\\
\Delta^{j+1} f(u,v)\big[v^{[j+1]}\big] &\coloneqq \Delta^j f(u,v)\big[v^{[j]}\big] - \frac{1}{j!} D^j f(u)\big[v^{[j]}\big],
\end{aligned}$$

where $Df(u)$ is the Fréchet derivative of $f$ around $u$, $D^2 f(u)$ is the second Fréchet derivative of $f$ around $u$, and so on. With this, so long as $f$ is $l$ times Fréchet differentiable and thus each $\Delta^j f(u,v)\big[v^{[j]}\big]$ defined for $j = 1, \ldots, l+1$, then we can write a Taylor series expansion of $f$ of the form

$$f(u+v) = \sum_{j=0}^{k} \Big[D^j f(u)\big[v^{[j]}\big]\Big] + \Delta^{k+1} f(u,v)\big[v^{[k+1]}\big] \tag{A.1}$$

for any $k \leq l$.

## A.1 Evolution operators

We consider an arbitrary, but in general nonlinear, system of $n$ differential equations in $d$ spatial dimensions and time, specifically $x \in \Omega \subset \mathbb{R}^d$, $t \in (t_a, t_b) \subset \mathbb{R}_{\geq 0}$,

$$\partial_t q_i(u) + \partial_\alpha f_i^\alpha(u) = 0, \qquad i = 1, \ldots, n, \tag{A.2}$$

where summation over $\alpha = 1, \ldots, d$ is implied, We first write this as the vector equation

$$\partial_t q(u) + \partial_\alpha f^\alpha(u) = 0,$$

where $u : \Omega \times (t_a, t_b) \to \mathcal{U} \subset \mathbb{R}^n$ is the state variable, $q : \mathbb{R}^n \to \mathbb{R}^n$ is the vector of conserved quantities, and $f : \mathbb{R}^n \to \mathbb{R}^{n+d}$ is the flux. For simplicity we further rewrite this system as

$$\partial_\alpha f^\alpha(u) = 0 \tag{A.3}$$

by re-indexing $\alpha = 0, \ldots, d$.

We define the evolution operator by

$$\mathcal{E}^\tau(u_0) \coloneqq u_\tau,$$

where any $t$-subscript, as we have here, is understood to mean *evaluation* at $t$ throughout this appendix. So, the evolution map

$$\mathcal{E}^\tau : u(\cdot, 0) \mapsto u(\cdot, \tau)$$

is a map that takes initial data $u_0$ to the solution of (A.3) at time $\tau$, which, again, we denote $u_\tau$.

That is to say, $\mathcal{E}^{\bar{x}}(u_0)$ takes in initial data $u_0(t)$, evolves it nonlinearly from $x = 0$ to $x = \bar{x}$ according to (A.2), and returns $u(\bar{x}, t)$. If we perturb the initial data of this solution, $u_0(t)$, by some $v_0(t)$ assumed to be such that $(u + v)(x, t)$ is still an exact solution of (A.2), expect with the initial condition

$$(u + v)(0, t) = u_0(t) + v_0(t),$$

then we find that $v(x, t)$ must satisfy

$$\partial_t v + \partial_x \big(F(u + v) - F(u)\big) = 0. \tag{A.4}$$

For a sufficiently differentiable $F$, we can write this as

$$\partial_t v + \partial_x Df(u_0)[v] = o(||v||), \tag{A.5}$$

where $DF(u)$ is the Fréchet derivative of $F(u)$.

Assuming $\mathcal{E}^{\bar{x}}$ is Fréchet-differentiable at $u_0$, then by definition there exists a bounded linear map, which we label $D\mathcal{E}^{\bar{x}}(u_0)$, that satisfies

$$\mathcal{E}^{\bar{x}}(u_0 + U_0) - \mathcal{E}^{\bar{x}}(u_0) - D\mathcal{E}^{\bar{x}}(u_0)[U_0] = o\big(||U_0||\big). \tag{A.6}$$

This map, $D\mathcal{E}^{\bar{x}}(u_0)$, is the evolution operator for the linear approximation of (A.5),

$$\partial_t U + \partial_x Df(u_0)[U] = 0. \tag{A.7}$$

We use uppercase initial data like $U_0(t)$ for initial data that perturbs $u_0(t)$ to a solution satisfying

$$\partial_t(u + U) + \partial_x(u + U) = o(||U||),$$

in contract to lowercase initial data $v_0$ that perturbs $u_0(t)$ to a solution of (A.4), making $u + v$ an exact solution of the same problem. We note that it is

$D\mathcal{E}^{\bar{x}}(u_0)[\cdot]$ that is a linear operator, which is represent with brackets. Nonlinear mappings are indicated with parentheses, as in $\mathcal{E}^{\bar{x}}(\cdot)$ or $D\mathcal{E}^{\bar{x}}(\cdot)[U_0]$.

The solution $U$ satisfies

$$\mathcal{E}^{\bar{x}}(u_0 + U_0) - \mathcal{E}^{\bar{x}}(u_0) - U(x_0, t) = o\big(||U_0||\big),$$

and so we write $U = D\mathcal{E}^{\bar{x}}(u_0)[U_0]$ and refer to $U$ as the linearization of $u$ around $u_0$ at, or in the direction of, $U_0$. Both the operator $D\mathcal{E}^{\bar{x}}(u_0)$ and the system of differential equations (A.7) may be referred to as just the linearization, but if necessary some more specificity is required, then we refer to (A.7) as the linearization of the system (A.2). However, as we are solving an initial boundary value problem, it will usually be more useful to talk about the entire linearized problem, consisting of the system (A.7) together with the initial and boundary conditions

$$U(0,t) = U_0(t) \quad \text{and} \quad U \cdot n|_{\partial\Omega} = h(x), \quad U(x,T) = U(x,0).$$

# B Pre-iteration corrections

Before starting any such iterative process however, we begin with some explicit corrections to $p^{(0)}(x,0)$ to improve our starting position,

$$p_0^{(1)}(x) = p_0^{(0)}(x) + \sum_{j\neq k} b_j \frac{\phi_j}{\bar{\sigma}} = \bar{p} + (\alpha\bar{p})\frac{\phi_k}{\bar{\sigma}} + (\alpha^2\bar{p})\sum_{j\neq k} b_j \frac{\phi_j}{\bar{\sigma}}.$$

We want to choose the coefficients $b_j$ so that the $\mathcal{O}(\alpha^2\bar{p})$ terms in the residual (3.22), which is just the one term

$$\frac{1}{2}(\alpha^2\bar{p})D^2 f(\bar{p})\left[\frac{\phi_k}{\bar{\sigma}}, \frac{\phi_k}{\bar{\sigma}}\right], \tag{B.1}$$

vanish, but finding out exactly what such choices for these coefficients will require some work. The second Fréchet derivative $D^2 f(\bar{p})$ in the quadratic term (B.1) corresponds to evolving $P^{[1,2]}(x,0) = 0$ and $U^{[1,2]}(x,0) = 0$ according to the system of the form (3.21), with $\tilde{p} = \bar{p}, \tilde{u} = 0$, which we write as

$$\begin{cases} \partial_t U^{[1,2]} + \partial_x P^{[1,2]} = 0 \\ \bar{\sigma}^2 \partial_t P^{[1,2]} + \partial_x U^{[1,2]} = v_{pp}(\bar{p}, s)\partial_t \left(P^{[1]} P^{[2]}\right), \end{cases} \tag{B.2}$$

with $P^{[1]}$ and $P^{[2]}$ evolving according to the linearization around the quiet state

$$\begin{cases} \partial_t U^{[i]} + \nabla P^{[i]} = 0 \\ \bar{\sigma}^2(s)\partial_t P^{[i]} - \nabla \cdot U^{[i]} = 0 \end{cases}$$

from whatever initial data is fed to $D^2 f(\bar{p})$ for each, which we have previously solved. Although (B.1) has $P_0^{[1]}$ and $P_0^{[2]}$ both equal to the $k$-mode $\frac{\phi_k}{\bar{\sigma}}$, which means we could directly calculate

$$D^2 f(\bar{p})\left[\frac{\phi_k}{\bar{\sigma}}, \frac{\phi_k}{\bar{\sigma}}\right],$$

but it will prove far more useful later on though if we instead calculate the more general

$$D^2 f(\bar{p}) \left[ \frac{\phi_i}{\bar{\sigma}}, \frac{\phi_j}{\bar{\sigma}} \right]$$

and then plug in $i = k$ and $j = k$.

So, if $P_0^{[1]} = \frac{\phi_i}{\bar{\sigma}}$ and $P_0^{[2]} = \frac{\phi_j}{\bar{\sigma}}$, then our solution of the linearization (3.9) yields

$$P_0^{[1]} = \frac{\phi_i}{\bar{\sigma}} \cos(\lambda_i t), \qquad P_0^{[2]} = \frac{\phi_j}{\bar{\sigma}} \cos(\lambda_j t).$$

With this, (B.2) becomes

$$\begin{cases} \partial_t U^{[1,2]} + \nabla P^{[1,2]} = 0 \\ \bar{\sigma}^2(s) \partial_t P^{[1,2]} + \nabla \cdot U^{[1,2]} = v_{pp}(\bar{p}, s) \partial_t \left( \cos(\lambda_i t) \cos(\lambda_j t) \frac{\phi_i(x)}{\bar{\sigma}} \frac{\phi_j(x)}{\bar{\sigma}} \right) \end{cases}$$

for which we use Duhamel's principle to solve. Namely, if we can solve the auxiliary system

$$\begin{cases} \partial_t U^{\varepsilon} + \nabla P^{\varepsilon} = 0 \\ \bar{\sigma}^2(s) \partial_t P^{\varepsilon} - \nabla \cdot U^{\varepsilon} = 0 \end{cases} \tag{B.3}$$

with initial data

$$P^{\varepsilon}(x, \varepsilon) = \frac{1}{2} v_{pp}(\bar{p}, s) \Big[ \lambda_{i-j} \sin\left(\lambda_{i-j} r\right) + \lambda_{i+j} \sin\left(\lambda_{i+j} r\right) \Big],$$

where we have used $\lambda_{i \pm j} = \lambda_i \pm \lambda_j$, and $U^{\varepsilon}(x, \varepsilon) = 0$, then we can find our desired solution by

$$U^{[1,2]}(x, T) = \int_0^T U^{\varepsilon}(x, T) \, d\varepsilon. \tag{B.4}$$

We solve (B.3) in a similar fashion to (3.4). The eigenfunction basis for $P^{\varepsilon}$ and $U^{\varepsilon}$ are again $\{\frac{\phi_n}{\bar{\sigma}}\}$ and $\{\psi_n\}$, respectively,

$$P^{\varepsilon}(x, t) = \sum_l P_l^{\varepsilon}(t) \frac{\phi_l}{\bar{\sigma}}, \quad U^{\varepsilon}(x, t) = \sum_l U_l^{\varepsilon}(t) \psi_l.$$

Taking appropriate inner products of each equation in (B.3), we find the same independent system for component pair $P_l^{\varepsilon}, U_l^{\varepsilon}$,

$$\begin{cases} d_t P_l^{\varepsilon} + \lambda_j U_l^{\varepsilon} = 0 \\ d_t U_l^{\varepsilon} - \lambda_j P_l^{\varepsilon} = 0, \end{cases}$$

where the initial conditions are

$$P_l^{\varepsilon}(x, \varepsilon) = \frac{1}{2} v_{pp}(\bar{p}, s) \Big[ \lambda_{i-j} \sin\left(\lambda_{i-j} r\right) + \lambda_{i+j} \sin\left(\lambda_{i+j} r\right) \Big] \Gamma_{i,j,l}$$

and $U_l^{\varepsilon}(x, \varepsilon) = 0$. We denote by $\Gamma_{i,j,l}$ the $\frac{\phi_l}{\bar{\sigma}}$ component of the product $\frac{\phi_i}{\bar{\sigma}} \frac{\phi_j}{\bar{\sigma}}$,

$$\Gamma_{i,j,l} = \left\langle \frac{\phi_i}{\bar{\sigma}} \frac{\phi_j}{\bar{\sigma}}, \frac{\phi_l}{\bar{\sigma}} \right\rangle_{w_1}.$$

So, again,

$$\frac{d}{dt}\left[R(-\lambda_l t)\begin{pmatrix} P_l^\varepsilon \\ U_l^\varepsilon \end{pmatrix}\right] = 0.$$

Since (B.3) specifies data for $P^\varepsilon$ and $U^\varepsilon$ at the time $t = \varepsilon$ instead of at $t = 0$, we simply evolve from $t = \varepsilon$ to $t = T$, finding

$$P_l^\varepsilon(T) = \cos\big(\lambda_l\,(T-\varepsilon)\big)P_l^\varepsilon(\varepsilon), \quad U_l^\varepsilon(T) = -\sin\big(\lambda_l\,(T-\varepsilon)\big)P_l^\varepsilon(\varepsilon).$$

Now that we know each of the components $U_l^\varepsilon$ from the auxiliary problem at $t = T$, we use (B.4) to find the $\psi_l$ component of $U^{[1,2]}$ at $t = T$. Doing so, we find

$$U_j^{[1,2]}(T) = -\tfrac{1}{2}\Gamma_{i,j,l}v_{pp}(\bar{p},s)\Bigg[\lambda_{i-j}\frac{\lambda_l\sin(\lambda_{i-j}T) - \lambda_{i-j}\sin(\lambda_l T)}{\lambda_{i-j}^2 - \lambda_l^2} + \lambda_{i+j}\frac{\lambda_l\sin(\lambda_{i+j}T) - \lambda_{i+j}\sin(\lambda_l T)}{\lambda_{i+j}^2 - \lambda_l^2}\Bigg] \quad \text{(B.5)}$$

Expanding $f\big(p_0^{(1)}\big)$ around $\bar{p}$, and again immediately simplifying with $f(\bar{p}) = 0$ and $Df(\bar{p})\left[\frac{\phi_k}{\bar{\sigma}}\right] = 0$, we have

$$f\big(p_0^{(1)}\big) = (\alpha^2\bar{p})\sum_{j\neq 0,k} b_j Df(\bar{p})\left[\frac{\phi_j}{\bar{\sigma}}\right] + \frac{1}{2}(\alpha\bar{p})^2 D^2 f(\bar{p})\left[\frac{\phi_k}{\bar{\sigma}},\frac{\phi_k}{\bar{\sigma}}\right] + \mathcal{O}(\alpha^3). \quad \text{(B.6)}$$

Our goal being to only correct the leading order $\mathcal{O}(\alpha^2\bar{p})$ term in the error of $p_0^{(0)}$, we drop any higher order terms and write

$$\sum_{j\neq 0,k} b_j Df(\bar{p})\left[\frac{\phi_j}{\bar{\sigma}}\right] = -\frac{1}{2}\bar{p}\,D^2 f(\bar{p})\left[\frac{\phi_k}{\bar{\sigma}},\frac{\phi_k}{\bar{\sigma}}\right].$$

Thus, we set

$$b_j = -\frac{\bar{p}\,\Gamma_{k,k,j}}{2Df(\bar{p})\left[\frac{\phi_j}{\bar{\sigma}}\right]} D^2 f(\bar{p})\left[\frac{\phi_k}{\bar{\sigma}},\frac{\phi_k}{\bar{\sigma}}\right].$$

Using our calculation of $U_j^{[1,2]}(T) = D^2 f(\bar{p})\left[\frac{\phi_k}{\bar{\sigma}},\frac{\phi_k}{\bar{\sigma}}\right]$ (B.5), this becomes

$$b_j = \frac{\lambda_k v_{pp}(\bar{p},s)\bar{p}\,\Gamma_{k,k,j}}{4\sin(\lambda_j T)}\left(\frac{1}{2\lambda_k+\lambda_j} + \frac{1}{2\lambda_k-\lambda_j}\right),$$

a choice of coefficients that ensures we improve the error $f\big(p_0^{(0)}\big) = \mathcal{O}(\alpha^2\bar{p})$

$$f\big(p_0^{(1)}\big) = \mathcal{O}\left(\alpha^3\bar{p}\right).$$

This improvement does not come with much of a computational cost, especially in comparison to the cost associated with a step of the quasi-Newton method we will define next.

From this improved solution, all further corrections are found numerically, but one could now, at least theoretically, design a second step explicit step

$$p_0^{(2)} = p_0^{(1)} + (\alpha^3\bar{p})\sum_{j\neq k} c_j \frac{\phi_j}{\bar{\sigma}},$$

where the $c_j$ coefficients are chosen so as to exactly cancel out the leading order $\mathcal{O}(\alpha^3\bar{p})$ term in $f\big(p_0^{(1)}\big)$ so that $f\big(p_0^{(2)}\big) = \mathcal{O}(\alpha^4\bar{p})$. However, the endeavor that is continuing on in this way already borders on the masochistic. If we expand (B.6) further, we find

$$f\big(p_0^{(1)}\big) = \frac{1}{6}(\alpha^3\bar{p})D^3(\bar{p})\left[\frac{\phi_k}{\bar{\sigma}}, \frac{\phi_k}{\bar{\sigma}}, \frac{\phi_k}{\bar{\sigma}}\right] + (\alpha^2\bar{p})\sum_{j\neq k} b_j D^2(\bar{p})\left[\frac{\phi_k}{\bar{\sigma}}, \frac{\phi_j}{\bar{\sigma}}\right] + \mathcal{O}(\alpha^4\bar{p}).$$

So these coefficients would be found by similarly dropping the higher order terms and setting

$$\sum_{j\neq k} c_j \frac{\phi_j}{\bar{\sigma}} = -\frac{1}{6}(\alpha^3\bar{p})D^3(\bar{p})\left[\frac{\phi_k}{\bar{\sigma}}, \frac{\phi_k}{\bar{\sigma}}, \frac{\phi_k}{\bar{\sigma}}\right] - (\alpha^2\bar{p})\sum_{j\neq k} b_j D^2(\bar{p})\left[\frac{\phi_k}{\bar{\sigma}}, \frac{\phi_j}{\bar{\sigma}}\right].$$

The second linearization terms $D^2 f(\bar{p})\big[\frac{\phi_k}{\bar{\sigma}}, \frac{\phi_j}{\bar{\sigma}}\big]$ where $j \neq k$ are again found by solving (B.2) but now with $P^{[1]}(x,0) = \frac{\phi_k}{\bar{\sigma}}$ and $P^{[2]}(x,0) = \frac{\phi_j}{\bar{\sigma}}$. So for each of these terms we must solve

$$\begin{cases} \partial_t U^{[1,2]} + \nabla P^{[1,2]} = 0 \\ \bar{\sigma}^2(s)\partial_t P^{[1,2]} + \nabla \cdot U^{[1,2]} = v_{pp}(\bar{p},s)\partial_t\left(\cos(\lambda_k t)\cos(\lambda_j t)\frac{\phi_k(x)\phi_j(x)}{\bar{\sigma}^2}\right) \\ P^{[1,2]}(x,0) = 0,\ U^{[1,2]}(x,0) = 0, \end{cases}$$

which can again be done with Duhamel's principle.

The $D^3(\bar{p})\big[\frac{\phi_k}{\bar{\sigma}}, \frac{\phi_k}{\bar{\sigma}}, \frac{\phi_k}{\bar{\sigma}}\big]$ term requires solving the third linearization. To find the third linearization, we just linearize the above system in a third direction $P^{[3]}, U^{[3]}$, after which we set

$$P^{[1]}(x,0) = P^{[2]}(x,0) = P^{[3]}(x,0) = \frac{\phi_k}{\bar{\sigma}}$$

and solve for $U^{[1,2,3]}$. In this case, Duhamel's principle leads to the system

$$\begin{cases} \partial_t U^{\varepsilon} + \nabla P^{\varepsilon} = 0 \\ \bar{\sigma}^2(s)\partial_t P^{\varepsilon} - \nabla \cdot U^{\varepsilon} = 0 \end{cases}$$

with initial conditions $U^{\varepsilon}(x,\varepsilon) = 0$ and

$$\begin{aligned} P^{\varepsilon}(x,\varepsilon) = v_{pp}(\bar{p},s)\partial_t\Big[P^{[1,2]}P^{[3]} + P^{[2,3]}P^{[1]} + P^{[3,1]}P^{[2]}\Big|_{t=\varepsilon} \\ v_{ppp}(\bar{p},s)\partial_t\Big[P^{[1]}P^{[2]}P^{[3]}\Big|_{t=\varepsilon}, \end{aligned} \tag{B.7}$$

where $P^{[1]}$, $P^{[2]}$, $P^{[3]}$ are solutions of the first linearization and $P^{[1,2]}$, $P^{[2,3]}$, $P^{[3,1]}$ are solutions of the second linearization. It is at this point that we remark that while having an explicit formula for correcting our solution is computationally fast, solving for $U_j^{[1,2,3]}(T)$ and then the correction coefficients $c_j$ in this way is, at the very least, overly tedious. Furthermore, doing so at a general $n$th step for $U_j^{[1,2,\ldots,n]}(T)$ to find coefficients to correct the $\mathcal{O}(\alpha^n)$ terms is just unfeasible.

## C 2D polar Fourier transforms

Here we provide more details on the relationship between the polar decomposition coefficients of a function on a disc and the Cartesian decomposition coefficients of the same function zero padded to the square. To start, the 2D Fourier transform of a function $f(\mathbf{r})$ expressed in Cartesian coordinates, $\mathbf{r} = (x, y)$, is written $F(\mathbf{k})$, where $\mathbf{k} = (k_x, k_y)$. The forward transform is given bay

$$F(\mathbf{k}) = F(k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i\mathbf{k}\cdot\mathbf{r}} \, dx \, dy,$$

and the inverse is given by

$$f(\mathbf{r}) = f(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(k_x, k_y) e^{i\mathbf{k}\cdot\mathbf{r}} \, dk_x \, dk_y.$$

We can consider polar versions of both the spatial coordinates $\mathbf{r} = (r, \theta)$ and the frequency coordinates $\mathbf{k} = (k, \psi_k)$, where $r, \theta$ are related to $x, y$ by

$$r^2 = x^2 + y^2, \qquad \theta = \arctan\left(\frac{y}{x}\right)$$

and $k, \psi_k$ are again related to $k_x, k_y$ by

$$k^2 = k_x^2 + k_y^2, \qquad \psi_k = \arctan\left(\frac{k_y}{k_x}\right).$$

A function $f(\mathbf{r})$ expressed in polar coordinates can be expanded as

$$f(\mathbf{r}) = f(x, y) = \sum_{n=-\infty}^{\infty} f_n(r) e^{in\theta}, \tag{C.1}$$

where the coefficients $f_n(r)$ are given by

$$f_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(r, \theta) e^{-in\theta} \, d\theta. \tag{C.2}$$

A function $F(\mathbf{k})$ expressed in polar coordinates can be similarly expanded as

$$F(\mathbf{k}) = F(k, \psi_k) = \sum_{-\infty}^{\infty} F_n(k) e^{in\psi_k},$$

where the coefficients $F_n(k)$ are given by

$$F_n(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\psi_k, k)e^{-in\psi_k}\, d\psi_k.$$

We find that, importantly, $F_n(k)$ is *not* the 1D Fourier transform of $f_n(r)$. One can show that the two coefficients are instead related by

$$F_n(k) = 2\pi i^{-n}\int_0^{\infty} f_n(r)J_n(kr)\, rdr,$$

where $J_n(r)$ is an $n$th order Bessel function of the first kind. We can thus write this in terms of the $n$th order Hankel transform $\mathbb{H}_n\{\cdot\}$,

$$F_n(k) = 2\pi i^{-n}\mathbb{H}_n\big(f_n(r)\big), \tag{C.3}$$

and so the inverse transform is written

$$f_n(r) = \frac{i^n}{2\pi}\mathbb{H}_n\big(F_n(k)\big).$$

## C.1 Infinite transform

We start with the case of a 2D function $f(\mathbf{r})$ defined on the entire plane. Any such function can be expanded in terms of the cylindrical wave functions

$$\Psi_{n,k}(r,\theta) = \sqrt{k}J_n(kr)\Theta_n(\theta).$$

Here, $\Theta_n(\theta)$ are the basis functions for the angular part of $\Psi_{n,k}$,

$$\Theta_n(\theta) = \frac{1}{2\pi}e^{in\theta}. \tag{C.4}$$

This expansion reads

$$f(\mathbf{r}) = f(r,\theta) = \int_0^{\infty}\sum_{n=-\infty}^{\infty} P_{n,k}\Psi_{n,k}(r,\theta)\, kdk, \tag{C.5}$$

where $P_{n,k}$, referred to as the polar Fourier coefficients, are given by

$$P_{n,k} = \int_0^{2\pi}\int_0^{\infty} f(r,\theta)\Psi^*_{n,k}(r,\theta)\, rdrd\theta.$$

These are the decomposition coefficients that we would like to avoid calculating via costly numerical integration methods.

Wang et al. [7] provide one way of avoiding such expensive decomposition methods by deriving a relationship between the function's polar Fourier coefficients and Cartesian Fourier coefficients. This has potential to be useful because whether working with MATLAB or Python, there is no standard "fast polar Fourier transform" one can call, but such an explicit relationship would

allow one to use the FFT to calculate the Cartesian coefficients and then find the polar coefficients from those. The basis functions for the Cartesian Fourier transform represent a plane wave,

$$\frac{1}{2\pi}e^{i\mathbf{r}\cdot\mathbf{k}},$$

which we can expand in terms of $\Psi_{n,k}$ as

$$\begin{aligned}\frac{1}{2\pi}e^{i\mathbf{r}\cdot\mathbf{k}} &= \frac{1}{2\pi}e^{irk\cos(\theta-\psi_k)} \\ &= \sum_{n=-\infty}^{\infty} i^n \frac{1}{2\pi} J_n(kr) e^{in(\theta-\psi_k)} \\ &= \sum_{n=-\infty}^{\infty} \frac{i^n}{\sqrt{2\pi k}} e^{-in\psi_k} \Psi_{n,k}(r,\theta). \end{aligned} \tag{C.6}$$

In addition to plane waves being decomposable into cylindrical waves of the same wavenumber, it can be shown that, likewise, cylindrical waves can be built by superpositions of plane waves of the same wave number,

$$\Psi_{n,k}(r,\theta) = \int_0^{2\pi} \frac{(-i)^m}{\sqrt{2\pi k}} e^{in\psi_k} \left(\frac{1}{2\pi}e^{ikr\cos(\theta-\psi_k)}\right) d\psi_k.$$

Just as we were able to expand $f(\mathbf{r})$ in terms of the cylindrical wave functions $\Psi_{n,k}$, we can of course expand $f(\mathbf{r})$ in terms of plane waves. We denote the Cartesian Fourier coefficients by $C_{\psi_k,k}$, where the subscript is written $\psi_k, k$ to indicate that both indices take continuous values. With this, we can expand $f(\mathbf{r})$ as

$$\begin{aligned} f(r,\theta) &= \int_0^{\infty}\int_0^{2\pi} C_{\psi_k,k}\left(\frac{1}{2\pi}e^{i\mathbf{r}\cdot\mathbf{k}}\right) k d\psi_k dk \\ &= \int_0^{\infty}\int_0^{2\pi} C_{\psi_k,k}\left(\frac{1}{2\pi}e^{ikr\cos(\theta-\psi_k)}\right) k d\psi_k dk. \end{aligned}$$

Using (C.6) here allows us to write

$$f(r,\theta) = \int_0^{\infty} \sum_{n=-\infty}^{\infty} \left[\frac{i^m}{\sqrt{k}}\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} C_{\psi_k,k} e^{-in\psi_k}\, d\psi_k\right] \Psi_{n,k}(r,\theta)\, k dk.$$

Wang et al. thus find $P_{n,k}$ to be the Fourier coefficient of $C_{\psi_k,k}$ for $\psi_k$ a variable,

$$P_{n,k} = \frac{i^m}{\sqrt{k}}\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} C_{\psi_k,k} e^{-in\psi_k}\, d\psi_k.$$

## C.2 Finite transform

We now move to the more pertinent case of a space limited function. Specifically, we consider the case we faced in section 4, a function $f(\mathbf{r})$ that is nonzero only

on $B_0^\nu$, the disc of radius $\nu$ centered on the origin,

$$\operatorname{supp}\big(f(\mathbf{r})\big) = \{\mathbf{r} \,|\, r \leq \nu\}.$$

For a transform on this finite region, we must use basis functions that are supported on this same region. These basis functions, which we denote by $\Psi_{n,m}^\nu$, are defined by

$$\Psi_{n,m}^\nu(r,\theta) = R_{n,m}^\nu(r)\Theta_n(\theta),$$

where $\Theta_n(\theta)$ is again the angular basis function (C.4) and $R_{n,m}^\nu$ is

$$R_{n,m}^\nu(r) = \frac{1}{\sqrt{N_m^{(n)}}} J_n(k_{n,m}r) \tag{C.7}$$

for $r \leq \nu$ and 0 otherwise. In the above, $k_{n,m}$ are values determined by the boundary condition, with each family $k_{n,1} < k_{n,2} < \cdots < k_{n,m} < \cdots$ being nonnegative solutions of

$$J_n(k\nu)\cos(\beta) - k\nu J_n'(k\nu)\sin(\beta), \tag{C.8}$$

for some specified choice of the parameter $\beta \in [0,\pi)$. Additionally, the $N_m^{(n)}$ term in (C.7) is just a normalization constant,

$$N_m^{(n)} = \int_0^\nu \big(J_n(k_{n,m}r)\big)^2 \, r dr.$$

The cylindrical wave functions in the previous infinite case had subscripts $n, k$, which was to indicate that while the first index, $n$, took discrete values, the second index, $k$, took continuous values. This led us to expand a function $f(\mathbf{r})$ on the plane in terms of $\Psi_{n,k}$ (C.5). The subscripts of our basis functions now being $n, m$ is then, naturally, to indicate that both indices take discrete values. So, a function $f(\mathbf{r})$ on the disc, which we can write

$$f(r,\theta) = \sum_{\mathbf{k}_0} C_{\mathbf{k}_0} \frac{1}{\sqrt{A}} e^{i\mathbf{r}\cdot\mathbf{k}_0} \tag{C.9}$$

with $C_{\mathbf{k}_0}$ the Cartesian Fourier coefficients of $f(\mathbf{r})$ zero padded to the rectangle enclosing the disc and $A$ the area of said rectangle, is expanded in $\{\Psi_{n,m}^\nu\}$ as

$$f(r,\theta) = \sum_{n,m} P_{n,m}\Psi_{n,m}^\nu(r,\theta). \tag{C.10}$$

We would like to again find some relationship between the polar coefficients $P_{n,m}$ and the Cartesian coefficients $C_{\mathbf{k}_0}$. In a similar fashion to how the relationship between these coefficients was found in the infinite case, we start by

expanding a plane wave in our discrete cylindrical wave functions $\{\Psi^{\nu}_{n,m}\}$,

$$\begin{aligned} e^{i\mathbf{r}\cdot\mathbf{k}} &= e^{irk\cos(\theta-\psi_k)} \\ &= \sum_m i^m \sum_n \left[\int_0^{\nu} R^{\nu}_{n,m}(r) J_m(kr)\, r dr\right] R^{\nu}_{n,m}(r) e^{im(\theta-\psi_k)} \\ &= \sum_{n,m} i^m \sqrt{2\pi} \left[\int_0^{\nu} R^{\nu}_{n,m}(r) J_m(kr)\, r dr\right] e^{-im\psi_k} \Psi_{n,m}(r,\theta). \end{aligned} \tag{C.11}$$

But here we face something that was not a factor in the infinite case, namely needing to specify what boundary condition the basis functions satisfy. It seems most of the literature on this subject focuses on the zero boundary condition, $R^{\nu}_{n,m}(\nu) = 0$, in which case we substitute

$$\int_0^{\nu} R^{\nu}_{n,m}(r) J_m(kr)\, r dr = (-1)^m \sqrt{2} k_{n,m} \frac{J_n(k\nu)}{k^2 - k^2_{n,m}}$$

into (C.11) to find

$$e^{i\mathbf{r}\cdot\mathbf{k}} = \sum_{n,m} (-1)^m i^n 2\sqrt{\pi} k_{n,m} \frac{J_n(k\nu)}{k^2 - k^2_{n,m}} e^{-in\psi_k} \Psi^{\nu}_{n,m}(r,\theta).$$

Using the above expression with (C.9) and (C.10), we find

$$P_{n,m} = \sum_{\mathbf{k}_0} \rho(\mathbf{k}_0; n, m)$$

with

$$\rho(\mathbf{k}_0; n, m) = (-1)^m i^n \frac{2\sqrt{\pi}}{\sqrt{A}} k_{n,m} \frac{J_n(k_0\nu)}{k_0^2 - k^2_{n,m}} e^{-in\psi_{k_0}}.$$

We are, however, not overly concerned with the case of the zero boundary condition, our problem instead requiring the zero derivative boundary condition,

$$\frac{d}{dr} R^{\nu}_{n,m}(r)\bigg|_{r=\nu} = 0.$$

With the zero derivative boundary condition determining the $k_{n,m}$'s, we instead can find

$$\int_0^{\nu} R^{\nu}_{n,m}(r) J_m(kr)\, r dr = (-1)^m \sqrt{2} k_{n,m} \frac{k}{\sqrt{k^2_{n,m}\nu^2 - n^2}} \frac{J'_n(k\nu)}{k^2 - k^2_{n,m}}.$$

Substituting this expression into (C.11) now leads us to find

$$P_{n,m} = (-1)^m i^n \frac{2\sqrt{\pi}}{\sqrt{A}} k_{n,m} \frac{k}{\sqrt{k^2_{n,m}\nu^2 - n^2}} \sum_{\mathbf{k}_0} \frac{k_0 J'_n(k_0\nu)}{k_0^2 - k^2_{n,m}} e^{-in\psi_{k_0}} C_{\mathbf{k}_0}. \tag{C.12}$$

So, we have another possible method of calculating polar Fourier coefficients: zero padding $f(\mathbf{r})$ to the rectangle enclosing the disc, calculating the Cartesian Fourier coefficients $C_{\mathbf{k}_0}$ with the usual 2D FFT, and then applying (C.12). However, there remains one additional method of polar decomposition in this dissertation, an algorithm that is actually specifically suited to the transform at hand, rather than a method by which to use the usual FFT despite it not being easily applicable to the problem.

## C.3 Discrete transform

This subsection of the appendix on 2D polar Fourier transforms primarily follows Baddour's development of a discrete 2D polar Fourier transform [2]. We start with a function $f(\mathbf{r})$ limited, again, to the disc of radius $\nu$, which we can still expand as (C.1) on the disc, with $f_n(r)$ still given by (C.2). These Fourier coefficients, the $f_n(r)$'s, can be expanded as a Fourier-Bessel series,

$$f_n(r) = \begin{cases} \sum_{m=1}^{\infty} P_{n,m} J_n(k_{n,m} r) & r < \nu \\ 0 & r \geq \nu, \end{cases}$$

where the $k_{n,m}$'s come from a particular choice of boundary condition of the form (C.8).

If $k_{m,n}$ is the $m$th zero of $J_n(r)$, corresponding to (C.8) with $\beta = 0$, the Fourier-Bessel series coefficients $P_{n,m}$ are found to be given by

$$P_{n,m} = \frac{2}{\nu^2 J_{n+1}^2(k_{n,m}\nu)} \int_0^{\nu} f_n(r) J_n(k_{n,m} r)\, r dr,$$

but a set of Bessel functions such that $J_n(k_{n,m}\nu) = 0$ will not be applicable in the case of the disc earlier. [2] For our use, we will instead opt to expand $f_n(r)$ as

$$f_n(r) = \begin{cases} \sum_{m=1}^{\infty} P_{n,m} R_{n,m}(r) & r < \nu \\ 0 & r \geq \nu, \end{cases} \tag{C.13}$$

where $R_{n,m}(r)$ is given by (C.7) and the $k_{n,m}$'s come from (C.8) with $\beta = \frac{\pi}{2}$, so

$$k_{n,m} = \frac{j'_{n,m}}{\nu}$$

with $j'_{n,m}$ the $m$th solution of

$$J'_n(r) = 0.$$

Referring to [1], we find the Fourier-Bessel series coefficients are now given by

$$\begin{aligned} P_{n,m} &= \int_0^{\nu} f_n(r) R_{n,m}(r)\, r dr \\ &= \frac{\sqrt{2} k_{n,m}}{J_n(k_{n,m}\nu)\sqrt{k_{n,m}^2\nu^2 - n^2}} \int_0^{\nu} f_n(r) J_n(k_{n,m} r)\, r dr \\ &= \frac{\sqrt{2} k_{n,m}}{J_n(k_{n,m}\nu)\sqrt{k_{n,m}^2\nu^2 - n^2}} \int_0^{\infty} f_n(r) J_n(k_{n,m} r)\, r dr. \end{aligned} \tag{C.14}$$

The last step is just rewriting the integral to be over $[0,\infty)$ by the fact that $f(\mathbf{r})$ is zero for $r > \nu$. We make this change because then with (C.3) we can write

$$P_{n,m} = \frac{k_{n,m} i^n}{\sqrt{2}\pi J_n(k_{n,m}\nu)\sqrt{k_{n,m}^2\nu^2 - n^2}} F_n(k_{n,m}). \tag{C.15}$$

So, we can rewrite the expression in (C.13) for $f_n(r)$, $0 \le r \le \nu$ as

$$f_n(r) = \frac{i^n}{\sqrt{2}\pi} \sum_{m=1}^{\infty} \frac{k_{n,m}}{J_n(k_{n,m}\nu)\sqrt{k_{n,m}^2\nu^2 - n^2}} J_n(k_{n,m}r) F_n(k_{n,m}\nu) \tag{C.16}$$

Evaluating this expression at $r_{n,m'} = \frac{k_{n,m'}}{k_{n,N_1}}\nu$ for any $N_1 \in \mathbb{N}^+$ and $m' < N_1$ yields

$$f_n\Big(\frac{k_{n,m'}}{k_{n,N_1}}\nu\Big) = \frac{i^n}{\sqrt{2}\pi} \sum_{m=1}^{\infty} \frac{k_{n,m}}{J_n(k_{n,m})\sqrt{k_{n,m}^2\nu^2 - n^2}} J_n\Big(\frac{k_{n,m}k_{n,m'}}{k_{n,N_1}}\Big) F_n(k_{n,m}\nu),$$

an exact expression since we are still summing over infinite $m$.

We now must make the assumption that the function $f(\mathbf{r})$ is not space-limited but band limited as well, meaning $F_n(k_{n,m}) = 0$ for all $m > N_1$, although Baddour does also allow for the case of a function being *effectively* band-limited, where $F_n(k_{n,m}) \approx 0$. This is equivalent to terminating the above series at $m = N_1$,

$$f_n\Big(\frac{k_{n,m'}}{k_{n,N_1}}\nu\Big) = \frac{i^n}{\sqrt{2}\pi} \sum_{m=1}^{N_1-1} \frac{k_{n,m}}{J_n(k_{n,m})\sqrt{k_{n,m}^2\nu^2 - n^2}} J_n\Big(\frac{k_{n,m}k_{n,m'}}{k_{n,N_1}}\Big) F_n(k_{n,m}\nu).$$

With this choice, the relationship between $f_n(r)$ and $F_n(k)$ can now be easily inverted. To do this we first multiply the above expression by $\frac{4J_n\left(\frac{k_{n,m'}k_{n,l}}{k_{n,N_1}}\right)}{k_{n,N_1}^2\nu^2 J_{n+1}^2(k_{n,m'})}$,

$$\begin{aligned}
&\sum_{m'=1}^{N_1-1} f_n\Big(\frac{k_{n,m'}}{k_{n,N_1}}\nu\Big) \frac{4J_n\left(\frac{k_{n,m'}k_{n,l}}{k_{n,N_1}}\right)}{k_{n,N_1}^2\nu^2 J_{n+1}^2(k_{n,m'})} \\
&\qquad = \sum_{m,m'=1}^{N_1-1} \left[ \frac{i^n k_{n,m}}{\sqrt{2}\pi J_n(k_{n,m})\sqrt{k_{n,m}^2\nu^2 - n^2}} \frac{4J_n\left(\frac{k_{n,m}k_{n,m'}}{k_{n,N_1}}\right) J_n\left(\frac{k_{n,m'}k_{n,l}}{k_{n,N_1}}\right)}{k_{n,N_1}^2\nu^2 J_{n+1}^2(k_{n,m'})} F_n(k_{n,m}\nu) \right]
\end{aligned}$$

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